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CANONICAL STRUCTURE OF YANG-MILLS THEORY[1]

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I consider the problem of defining canonical coordinates and momenta in pure Yang-Mills theory, under the condition that Gauss' law is identically satisifed. This involves among other things particular boundary conditions for certain dependent variables. These boundary conditions are not postulated a priori, but arise as consistency conditions related to the equations of motion. It is shown that the theory indeed has a canonical structure, provided one uses a special gauge condition, which is a natural generalisation to Yang-Mills theory of the Coulomb gauge condition in electrodynamics. The canonical variables and Hamiltonian are explicitly constructed. Quantisation of the theory is briefly discussed.

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1 Introduction

In this paper I present a novel solution to an old problem in Yang-Mills theory [1], namely the problem of formulating the Yang-Mills equations of motion as a set of canonical Hamiltonian equations. The canonical formulation of Yang-Mills theory has been considered as a neccessary prerequisite for the development of a consistent quantisation procedure of this theory, as emphasised e.g. in the textbook on gauge theory by Faddeev and Slavnov [2]. I refer to this textbook for an account of the state of the art in Yang-Mills theory around 1980. An account of more recent developments can be found e.g. in the textbook by Weinberg [3].

The early attempts to quantise Yang-Mills theory [4], [5] were based on the use of the Coulomb gauge condition, familiar from electrodynamics, in Yang-Mills theory. However, as shown by Gribov [6] in 1977, the Coulomb gauge is not a proper gauge condition in Yang-Mills theory. It should be noted here that the problems pointed out by Gribov concerning the use of the Coulomb condition, the so-called Gribov ambiguities, do not really matter in perturbative calculations.

Another possible approach to a canonical formalism in Yang-Mills theory [7], is based on the so-called Weyl gauge [8], $A_0 = 0$. However, in this formulation one simply disregards the

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Gauss law, and considers only the canonical Hamiltonian formulation of the remaining Yang-Mills equations, which is a rather simple problem. Gauss' law is then in the quantised version of this theory introduced by hand, as a condition on the states in the theory.

There has recently been an upsurge in interest in non-perturbative aspects of Yang-Mills theory, as witnessed e.g. by a paper by Faddeev and Niemi [9]. It is emphasised by Faddeev and Niemi in their paper, that we are still lacking theoretical understanding of low-energy Yang-Mills theory, i.e. the non-perturbative regime, which is relevant for studying questions such as colour confinement. The problem of confinement is definitely a question which calls for a proper quantum Yang-Mills theory. A proper canonical formulation of the (semiclassical) Yang-Mills theory may well be a necessary prerequisite for this, or at least an appropriate starting point.

The real confinement problem concerns the situation when quark fields are included in the formalism, but this extension is not an essential difficulty in the formulation presented here.

The canonical formulation developed in this paper differs from the Weyl gauge formulation in that Gauss' law is implemented identically (in principle). This has necessitated the introduction of a new gauge condition [10], which is a straightforward and natural generalisation to Yang-Mills theory of the ordinary Coulomb gauge condition in electrodynamics. The resulting canonical Yang-Mills theory, in its Schrödinger quantised form, resembles the corresponding theory based on the Weyl gauge in many ways, but is apparently not isomorphic to that theory.

2 Notation and Conventions

The basic variables in Yang-Mills theory [1] are the gauge field $G_{\mu\nu}$ and gauge potential A_μ, respectively. These quantities take values in a convenient (Hermitian) matrix representation of the Lie algebra of the gauge group G. The Lie algebra is defined by the structure constants $f_{ab}^{\ \ c}$ in the commutator algebra of the Hermitian matrix representatives T_a of the Lie algebra generators,

$$[T_a, T_b] = i f_{ab}^{\ \ c} T_c, \tag{1}$$

where appropriate summation over repeated Lie algebra indices $(a, b, c, ..., f)$ is understood. We assume G to be semisimple and compact. Then a positive definite Lie algebra metric can be given in terms of the following Killing form (h_{ab}),

$$h_{ab} = -f_{ab'}^{\ \ \ c'} f_{bc'}^{\ \ \ b'}. \tag{2}$$

The inner product (A, B) of any two Lie algebra valued quantites $A = A^a T_a$ and $B = B^a T_a$ is defined as follows,

$$(A, B) = h_{ab} A^a B^b. \tag{3}$$

The form h_{ab} and its inverse h^{ab} are used to lower and raise Lie algebra indices, respectively.

In the notation introduced so far, we write the gauge potential A_μ as follows,

$$A_\mu(x) = A_\mu^a(x) T_a, \tag{4}$$

where the argument x stands for a space-time point in Minkowski space. The gauge field $G_{\mu\nu}(A)$ is then given as follows,

$$G_{\mu\nu}(A) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) - ig[A_\mu(x), A_\nu(x)], \tag{5}$$

where g is an arbitrary nonvanishing real parameter, which is introduced for convenience. An alternative to the notation (5) is the following,

$$G^a_{\mu\nu}(A) = \partial_\nu A^a_\mu(x) - \partial_\mu A^a_\nu(x) + g f_{bc}{}^a A^b_\mu(x) A^c_\nu(x). \tag{6}$$

The remaining notation is fairly conventional. Greek letters $\mu, \nu, ...$ are spacetime indices which take values in the range $(0, 1, 2, 3)$. These indices are lowered (raised) with the standard diagonal Minkowski metric $g_{\mu\nu}$ $(g^{\mu\nu})$ with signature $(+, -, -, -)$. Latin indices from the middle of the alphabet $(k, \ell, ...)$ are used as space indices in the range $(1, 2, 3)$. Unless otherwise stated, repeated indices are always summed over, be they Lie algebra-, spacetime- or space indices.

I finally state the gauge transformation formulae for the gauge potential and field, respectively, in my notation. Let $\Omega(x)$ denote a general gauge transformation, the generic form of which is a unitary matrix of the following form,

$$\Omega(x) = \exp(ig\alpha^a(x)T_a), \tag{7}$$

where the functions $\alpha^a(x)$ are any sufficiently smooth real-valued functions on Minkowski space. The relevant gauge transformation formulae are as follows,

$$A_\mu(x) \xrightarrow{\Omega} A'_\mu(x) = \Omega^{-1}(x)A_\mu(x)\Omega(x) + \frac{i}{g}(\partial_\mu \Omega^{-1}(x))\Omega(x), \tag{8}$$

and

$$G_{\mu\nu}(A) \xrightarrow{\Omega} G_{\mu\nu}(A') = \Omega^{-1}(x) G_{\mu\nu}(A)\Omega(x). \tag{9}$$

3 The equations of motion and Gauss' law

The Yang-Mills action S, with a general potential A, which at this stage is not assumed to satisfy any particular gauge condition, is as follows,

$$S = -\frac{1}{4} \int d^4x (G_{\mu\nu}(A), G^{\mu\nu}(A)). \tag{10}$$

It is convenient to write the action (10) in terms of a Lagrangian L, although this treats time- and space indices on an unequal footing. The action S is the integral of a Lagrangian L in an appropriate time interval $[x^0_i, x^0_f]$. Thus,

$$S = \int_{x^0_i}^{x^0_f} dx^0 L, \tag{11}$$

where

$$L = -\frac{1}{2} \int_V d^3\mathbf{x} \left(G_{0k}(A), G^{0k}(A) \right) - \frac{1}{4} \int_V d^3\mathbf{x} \left(G_{k\ell}(A), G^{k\ell}(A) \right). \tag{12}$$

In the expression (12) for the Lagrangian L, the quantity V is some appropriate domain in $\mathbf{R}^3$, which yet has to be specified.

As is well known, requiring the action (10) to be stationary with respect to local variations of *all* the potential components A_μ, considered as independent quantities, yields the follwing field equations,

$$\nabla_\nu(A)G^{\mu\nu}(A) = 0 \ , \mu = 0, 1, 2, 3. \tag{13}$$

The "covariant gradient" $\nabla_\mu(A)$ used above in Eq. (13) is a convenient notion,

$$\nabla_\mu(A) \equiv \partial_\mu + ig\,[A_\mu, \quad], \tag{14}$$

which will be frequently used in what follows.

A perhaps more interesting system of equations would be obtained by coupling the gauge field to appropriate matter fields. Then the right hand side of Eq. (13) would be replaced by a co-variantly conserved matter current. Such an addition is not absolutely essential for the questions pursued in this paper, and will therefore not be contemplated further here.

The non-Abelian Gauss law is obtained from the equations (13) for $\mu = 0$. Expressed in terms of the potential A the non-Abelian Gauss law reads as follows,

$$\nabla_k(A)\nabla^k(A)A^0 - \nabla_k(A)\dot{A}^k = 0. \tag{15}$$

Considered as an equation which determines the Lie algebra valued potential component A_0, for given space components $\mathbf{A} = (A^1, A^2, A^3)$, the equation (15) is a system of *linear, elliptic* partial differential equations with the time variable $x^0 \in [x_i^0, x_f^0]$ acting as a parameter.

From now on it will be convenient to denote the time derivative of any quantity with a dot on top of that quantity, thus for example

$$\dot{A}_k(x) \equiv \partial_0 A_k(x). \tag{16}$$

The apparent equations of motion are obtained from the field equations (13) for $\mu \equiv k = 1, 2, 3$. These equations read as follows, separated into two groups of first-order equations for convenience,

$$\dot{A}^k = \nabla^k(A)A^0 - G^{0k}(A) \ , \dot{G}^{0k}(A) = \nabla_\ell(A)G^{k\ell}(A) - ig[A_0, G^{0k}(A)]. \tag{17}$$

I then return to Eq. (15), i.e. the Gauss' law constraint. In order to analyse the solvability of Eq. (15), one has to specify the *domain* V of the independent variables $\mathbf{x} = (x^1, x^2.x^3) \in \mathbf{R}^3$ in this equation.

In what follows, I assume that the domain V is a finite simply connected domain in $\mathbf{R}^3$ with a smooth boundary ∂V. An example of a possible domain V would be a ball B_R of radius R, centered at the origin of space coordinates, i.e.

$$B_R = \{\mathbf{x} | \mid \mathbf{x} \mid < R\}\,, \tag{18}$$

keeping in mind the possibility of taking the limit $R \to \infty$ at some appropriate stage in the development of the formalism.

The solution A_0 of Eq. (15), assuming the existence of a unique solution, is, in general, an x-dependent *functional* of *both* the space components $\mathbf{A}$ and their time derivatives $\dot{\mathbf{A}}$, i.e.

$$A_0 = A_0\left\{\mathbf{A}, \dot{\mathbf{A}}\right\}. \tag{19}$$

The question is then whether the Yang-Mills system, which is originally defined by the action (10), permits a canonical structure when the potential component A_0 is a solution to Gauss' law (15), i.e. when A_0 is given in terms of $\mathbf{A}$ and $\dot{\mathbf{A}}$ by the expression (19). This question can be analysed without specifying the actual functional form of the relation (19) in minute detail.

The Lagrangian of the Yang-Mills system, when the potential component A_0, is given by the relation (19), is obtained simply by inserting the solution (19) for A_0 into the Lagrangian (12) above. The resulting Lagrangian which will be called L_0, is then explicitly given as follows,

$$\begin{aligned} L_0 &= -\frac{1}{2}\int_V d^3\mathbf{x}\left(\nabla_k(A)A_0\left\{\mathbf{A},\dot{\mathbf{A}}\right\}-\dot{\mathbf{A}}_k,\nabla^k(A)A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\}-\dot{\mathbf{A}}^k\right) \\ &\quad -\frac{1}{4}\int_V d^3\mathbf{x}\left(G_{kl}(A),G^{kl}(A)\right). \end{aligned} \tag{20}$$

At this point it is appropriate to check whether the action principle involving the Lagrangian L_0 in (20) above reproduces the equations of motion (17). It is perfectly straightforward to verify the following result,

$$\begin{aligned} \delta\int_{x_i^0}^{x_f^0}dx^0 L_0 = &-\int_{x_i^0}^{x_f^0}dx^0\int_V d^3\mathbf{x}\left(\delta A_k,\nabla_0(A)(\nabla^k(A)A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\}-\dot{\mathbf{A}}^k)-\nabla_\ell G^{k\ell}(A)\right) \\ &-\int_{x_i^0}^{x_f^0}dx^0\int_{\partial V}d^2\sigma_k\left(\delta A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\},\nabla^k(A)A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\}-\dot{\mathbf{A}}^k\right). \end{aligned} \tag{21}$$

Here the boundary conditions for the solution A_0 to Gauss' law, i.e. the system of linear elliptic partial differential equations (15), come into play. The surface term in Eq. (21), if non-vanishing, destroys the variational principle which is supposed to lead to the equations of motion (17) with A_0 given by (19).

Using the interpretation of the domain V given above in Eq. (18) in the limiting case $R\to\infty$, one finds that the surface term above in Eq. (21) vanishes if

$$\lim_{R\to\infty}\int_{|\mathbf{x}|=R}d\Omega R^2\,(\delta A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\},\nabla^{(r)}(A)A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\}-\dot{\mathbf{A}}^{(r)})=0, \tag{22}$$

for all admissible variations of the independent generalized coordinates $\mathbf{A}$ and velocities $\dot{\mathbf{A}}$, respectively. In the equation above, the superscript (r) indicates the *radial* component.

It will now simply be assumed that Eq. (22) can be used as a boundary condition; an assumption which will be examined in some detail in a forthcoming paper [11]

Under these circumstances the variational principle

$$\delta\int_{x_i^0}^{x_f^0}dx^0 L_0 = 0, \tag{23}$$

leads to the following equations of motion,

$$\nabla_0(A)\left\{\nabla^k(A)A^0\left\{\mathbf{A},\dot{\mathbf{A}}\right\}-\dot{\mathbf{A}}^k\right\}-\nabla_\ell G^{k\ell}(A)=0, \tag{24}$$

as is evident from the relation (21). Needless to say, the equations (24) are nothing but the original equations of motion (17), albeit in a slightly disguised form, with A_0 given by the solution (19) to Gauss' law (15).

4 Canonical formulation of the equations of motion

4.1 First attempt at constructing canonical momenta

Using the Lagrangian (20) as a starting point, one may now attempt to construct a canonical description of the system defined by the equations of motion (24).

The formal definition of the canonical momentum P^a_k conjugate to the coordinate A^k_a is,

$$P^a_k(x^0, \mathbf{x}) \equiv \frac{\delta L_0}{\delta \dot{A}^k_a(x^0, \mathbf{x})} = \left(\nabla_k(A) A_0 \left\{\mathbf{A}, \dot{\mathbf{A}}\right\}\right)^a (x^0, \mathbf{x}) - \dot{A}^a_k(x^0, \mathbf{x}), \tag{25}$$

where the condition (22) has been used in the calculation of the functional derivative of L_0 in (25) above. In view of the fact that A_0 in Eq. (25) satisfies Gauss' law (15), one finds immediately from (25) that

$$\nabla^k(A) P_k(x^0, \mathbf{x}) \equiv 0. \tag{26}$$

Now one is supposed to be able to solve Eq. (25) for the generalized velocity $\dot{A}^a_k$ in terms of $\mathbf{A}$ and $\mathbf{P}$, respectively. But this is impossible, since, Eq. (25) can not be solved for the quantity Γ defined below,

$$\Gamma(x^0, \mathbf{x}) \equiv \nabla_k(A) \dot{A}^k(x^0, \mathbf{x}), \tag{27}$$

i.e. if one tries to derive an equation for the quantity Γ defined above, from Eq. (25), one gets a completely vacuous identity for this quantity, as a result of Eq. (26).

It would seem then, that the equations of motion (24) can not be written as a system of canonical first order equations at all. However this is actually not the case.

It turns out, that the quantity Γ defined above in (27), can be set to zero as a consequence of the freedom of choosing a gauge in Yang-Mills theory. To be precise, the condition

$$\nabla_k(A) \dot{A}^k(x^0, \mathbf{x}) = 0 \tag{28}$$

has recently been shown to be a *proper gauge condition*, called the generalized Coulomb gauge condition [10].

In what follows, it will be assumed that the generalized Coulomb gauge condition (28) is in force. Since the generalized velocities $\dot{A}^k$ then no longer are independent quantities, one cannot use the formula (25) as it stands for the construction of the canonical momentum variables. An alternative procedure that leads to a canonical formalism will be given below.

4.2 Canonical coordinates, momenta and Hamiltonian

It is now appropriate to summarise the various conditions on the generalized coordinates $\mathbf{A}$ and genralised velocities $\dot{\mathbf{A}}$, respectively, that have come out of the previous analysis.

In the first place, it is assumed, that the *generalized Coulomb gauge condition* (28) is in force. Then *Gauss' law* (15) simplifies, and takes the following form,

$$\nabla_k(A) \nabla^k(A) A^0 = 0 \, , \mathbf{x} \in V \subset \mathbf{R}^3. \tag{29}$$

Hence *the solution* A_0 to Gauss' law (29) is a functional of the generalized coordinates $\mathbf{A}$ *only*,

$$A_0 = A_0\{\mathbf{A}\}\,. \tag{30}$$

The Lagrangian for the system under the conditons detailed above, is recorded here for the sake of completeness. This Lagrangian, denoted by L_{00}, is nothing but the Lagrangian of Eq. (20), but with the solution (30) for A_0 in stead of (19). Explicitly,

$$\begin{aligned} L_{00} &= -\frac{1}{2}\int_V d^3\mathbf{x}\left(\nabla_k(A)A_0\{\mathbf{A}\} - \dot{\mathbf{A}}_k, \nabla^k(A)A^0\{\mathbf{A}\} - \dot{\mathbf{A}}^k\right) \\ &\quad -\frac{1}{4}\int_V d^3\mathbf{x}\left(G_{kl}(A), G^{kl}(A)\right). \end{aligned} \tag{31}$$

I will now implement the gauge conditions (28) as a *constraint*, by means of a Lie algebra, or rather matrix valued Lagrange multiplier field $C(x)$, which is used to modify the Lagrangian L_{00} in (31) in an appropriate way,

$$L_{00} \to L' = L_{00} + \int_V d^3\mathbf{x}(C(x), \nabla_k(A)\dot{A}^k). \tag{32}$$

The Lagrange multiplier technique as such is well known in general as well as in the particular case of gauge systems [12].

One can now make a direct transition to a Hamiltonian using the modified Lagrangian (32) above, in a manner described in the general case by Berezin [13]. The starting point is the familiar definition of canonical momentum variables π_k^a,

$$\pi_k^a(x^0,\mathbf{x}) \equiv \frac{\delta L'}{\delta \dot{A}_a^k(x^0,\mathbf{x})} = \left(\nabla_k(A)A_0\{\mathbf{A}\}\right)^a - \dot{A}_k^a - \left(\nabla_k(A)C\right)^a, \tag{33}$$

where x^0 is a fixed parameter with variable $\mathbf{x} \in V$. The argument $(x^0,\mathbf{x})$ is omitted in the last term in (33) above. Similar simplifications of notation will be used freely below, whenever expedient. The equations (33) above, *together* with the constraint equations (28) are now supposed to be solved for the quantities $\dot{A}_k^a$ and C^a in terms of the canonical coordinates A_a^k and momenta π_k^a, respectively.

Using Eqns. (28) and (29), one finds immediately from Eq. (33) that

$$-\nabla_k(A)\nabla^k(A)C = \nabla^k(A)\pi_k, \tag{34}$$

which, together with appropriate boundary conditions, defines the quantity C as an x-dependent functional of $\mathbf{A}$ and π, respectively,

$$C = C\{\mathbf{A},\pi\}(x^0,\mathbf{x}). \tag{35}$$

It turns out that the appropriate boundary conditions are as follows,

$$\lim_{R\to\infty}\int_{|\mathbf{x}|=R} d\Omega R^2(C(x), \nabla^{(r)}(A)C(x)) = 0, \tag{36}$$

where the superscript (r) again indicates the radial component. The boundary condition above guarantees the uniqueness of the solution $C(x)$ to Eq. (34), as can easily be proved [11].

One now straightforwardly expresses the generalized velocity in terms of coordinate- and momentum variables,

$$\dot{A}_k^a = (\nabla_k(A_0\{\mathbf{A}\} - C))^a - \pi_k^a. \tag{37}$$

The construction of the Hamiltonian H then proceeds in the usual way. The relation defining the Hamiltonian H is the following,

$$H = \int_V d^3\mathbf{x}(\pi_k, \dot{A}^k) - L_{00}, \tag{38}$$

where the quantity $\dot{A}^k$ ocurring in the expressions in (38) should be given in terms of canonical variables by the expression (37). It should be observed, that it is indeed the Lagrangian L_{00} which enters in the definition of the Hamiltonian H above, since at this stage the constraint (28) is an identity.

By straightforward calculation one finally obtains the Hamiltonian expressed in terms of canonical variables from the definition (38),

$$\begin{aligned} H &= -\frac{1}{2}\int_V d^3\mathbf{x}\left(\pi_k, \pi^k\right) + \frac{1}{4}\int_V d^3\mathbf{x}\left(G_{kl}(A), G^{kl}(A)\right) \\ &\quad + \int_V d^3\mathbf{x}\left(\pi_k, \nabla^k(A)A^0\{\mathbf{A}\}\right) + \frac{1}{2}\int_V d^3\mathbf{x}\left(\nabla_k(A)C, \nabla^k(A)C\right), \end{aligned} \tag{39}$$

where the quantity C is the appropriate solution to the system of linear elliptic partial differential equations (34), as discussed previously.

By functional differentiation of the Hamiltonian H in (39) one finds,

$$\dot{A}_k^a(x^0, \mathbf{x}) \equiv \frac{\delta H}{\delta \pi_a^k(x^0, \mathbf{x})} = (\nabla_k(A)(A_0\{\mathbf{A}\} - C))^a - \pi_k^a, \tag{40}$$

which agrees precisely with the expression (37) as it should.

It is much more intricate to obtain the second half of the Hamiltonian equations, which give the time evolution of the canonical momentum variables π_a^k. After a lengthy but fairly straightforward calculation, one finds that

$$\begin{aligned} \dot{\pi}_a^k(x^0, \mathbf{x}) \equiv -\frac{\delta H}{\delta A_k^a(x^0, \mathbf{x})} &= -ig[A_0\{\mathbf{A}\} - C, \nabla^k(A)C + \pi^k]_a \\ &\quad + \left(\nabla_\ell G^{k\ell}(A)\right)_a - ig[C, \nabla^k(A)A_0\{\mathbf{A}\}]_a. \end{aligned} \tag{41}$$

Needless to say, the pair of Hamiltonian equations of motion, Eqns. (40) and (41) are in perfect agreement with the Lagrangian equations of motion obtained from the variation principle,

$$\delta \int dx^0 L' = 0, \tag{42}$$

where the Lagrangian L' is given in Eq. (32).

We have thus finally arrived at a canonical Hamiltonian formalism for the Yang-Mills system, in the case when Gauss' law is supposed to be identically satisfied. As was seen above, this required, among other things, that a special gauge condition, called the generalized Coulomb gauge condition [10], was in force.

The interest in the canonical structure of Yang-Mills theory comes from the interest in quantising this theory, either by canonical operator methods or by functional integral methods. I hope to return to such questions in the near future.

Before concluding this subsection, it is appropriate to notice, that the canonical Hamiltonian equations (40) and (41) admit a *first integral*, i.e.

$$\partial_0(\nabla_k(A)\pi^k) = 0. \tag{43}$$

The quantity $\nabla_k(A)\pi^k$ is thus a constant of motion. This is related to the fact, that the description of the Yang-Mills system in terms of the canonical variables A_a^k and π_k^a introduced here, is not a description in terms of a minimal number of dynamical degrees of freedom. Also this circumstance will be discussed in more detail elesewhere [11].

4.3 Comparison with the Weyl gauge formalism

For completeness I present below the Hamiltonian formulation of Yang-Mills theory in the so-called Weyl gauge [8] $A_0 = 0$. This has been discussed with admirable clarity in a paper by Jackiw [7], to which I refer for details. This paper also discusses the Schrödinger quantisation of the Weyl gauge version of Yang-Mills theory.

The Yang-Mills Lagrangian in the case $A_0 = 0$, which will be called L_W here, is the following,

$$L_W = -\frac{1}{2}\int d^3\mathbf{x}(\dot{A}_k, \dot{A}^k) - \frac{1}{4}\int d^3\mathbf{x}(G_{k\ell}, G^{k\ell}). \tag{44}$$

The Lagrangian (44) describes a theory which is *not* the same as Yang-Mills theory, since Gauss' law is absent from the field equations following from the action principle with the expression (44) as Lagrangian. Gauss' law takes the following form in the present case,

$$\nabla_k(A)\dot{A}^k = 0. \tag{45}$$

The traditional way to analyse Yang-Mills theory in the Weyl gauge, is to proceed from the Lagrangian (44), disregarding Gauss' law to begin with.

Using the variables A_k^a and $\dot{A}_k^a$ as generalized coordinates and velocities, respectively, it is perfectly simple to derive a canonical Hamiltonian formalism for the system defined by the Lagrangian (44). The corresponding Hamiltonian H_W is,

$$H_W = -\frac{1}{2}\int d^3\mathbf{x}(\pi_k, \pi^k) + \frac{1}{4}\int d^3\mathbf{x}(G_{k\ell}, G^{k\ell}), \tag{46}$$

i.e. a simple sum of a kinetic term, depending on π only, and a potential, or interaction term, depending on A only.

The Hamiltonian H_W in Eq. (46) should now be compared with the Hamiltonian (39) derived in the previous subsection. Disregarding for the time being the slightly different meaning of the symbols in the two Hamiltonians in question, one observes that the Hamiltonian (39) contains interaction terms depending on the quantities A_0 and C respectively, which are completely absent from the Weyl gauge Hamiltonian (46).

A transition to quantum versions of the Hamiltonians above, by means of the fixed time Schrödinger quantisation rule,

$$\pi_k^a \to \hat{\pi}_k^a \equiv -i\frac{\delta}{\delta A_a^k}, \tag{47}$$

facilitates a comparison of the two Hamiltonians.

Interpreting the Hamiltonian (39) as a quantum operator $\hat{H}$ by means of the substitution (47) one is then invited to consider the following eigenvalue equation, in self-explanatory notation,

$$\hat{H}\Psi(A) = E\Psi(A), \tag{48}$$

which in principle should describe a glueball spectrum. There are pitfalls associated with the quantisation procedure briefly described above, which have to be understood and avoided before considering conclusions based on equations such as (48), as being ironclad.

In the Weyl gauge case one would have, similarily,

$$\hat{H}_W\Psi_W(A) = E\Psi_W(A). \tag{49}$$

At this stage one simply adds the hitherto omitted Gauss' law (45) as a condition on the states $\Psi_W(A)$,

$$\left(\frac{\partial}{\partial x^k}\frac{\delta}{\delta A_k^a(x^0,\mathbf{x})} - gf_{ab}{}^c A_k^b(x^0,\mathbf{x})\frac{\delta}{\delta A_k^c(x^0,\mathbf{x})}\right)\Psi_W(A) = 0. \tag{50}$$

It is now appropriate to recall the first integral (43) admitted by the (semiclassical) system described by the Hamiltonian H in (39). In the Schrödinger quantisation of that system, the relation (43) gives rise to a condition on the states $\Psi(A)$ of the same type as (50), although not necessarily identical to that relation. Even if one might reproduce exactly a condition of the form (50) in the Schrödinger quantised theory defined by the quantum Hamiltonian $\hat{H}$, by an appropriate choice of a constant of integration, one must still reckon with the fact that the Hamiltonian $\hat{H}$ definitely differs from the Hamiltonian $\hat{H}_W$ by the interaction terms involving the quantities A_0 and C in the (semiclassical) expression (39). These interaction terms, appeared as a result of the demand that Gauss' law should be satisfied identically. The variables A_0 and C in these intreraction terms had to satisfy certain very precise boundary conditions in order to ensure the consistency of the (semiclassical) Lagrangian and Hamiltonian formulation, as explained previously. Hence these interaction terms cannot be declared to be uneccesary or unmiportant. So, one is forced to conclude that the Weyl gauge quantum canonical formalism in Yang-Mills theory differs in substance from the corresponding formalism developed in this paper.

5 Summary and Conclusions

In this paper I have considered the problem of defining canonical coordinates and momenta in pure Yang-Mills theory, under the condition that Gauss' law is identically satisifed. Satisfying

Gauss' law in this context means solving a system of elliptic linear partial differential equations for the matrix valued gauge potential component A_0, with a fixed value of the time coordinate. Existence of solutions to these equations is not proved here, but assumed. The related question of uniqueness of solutions, which depends on boundary conditions, is only briefly touched upon.

The canonical structure of the theory requires the choice of a special gauge, called the generalized Coulomb gauge in this paper. This gauge, the existence of which is equivalent to the existence of appropriate solutions to a system of partial differential equations of the same type as encountered in connection with Gauss' law, does in principle not suffer from Gribov ambiguitites.

The canonical variables, coordinates and momenta, as well as the Hamiltonian, are explicitly constructed. This construction involves again a system of partial differential equations of the same type as encountered previously, but now with different boundary conditions, which are dictated by necessary consistency conditions for the existence of canonical Hamiltonian equations of motion.

Schrödinger quantisation of the theory is briefly discussed, and compared with the corresponding quantum theory in the Weyl gauge, $A_0 = 0$. It is concluded that the Schrödinger quantised Hamiltonian formulation of Yang-Mills theory developed in this paper, differs in substance from the corresponding Weyl gauge formulation.

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