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REALIZATIONS OF $SU(1,1)$ BY BOSON OPERATORS WITH APPLICATION TO PHASE STATES*

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A class of realizations of the abstract Lie algebra $su(1,1)$ in the basis (K_-, K_0, K_+) by one-mode boson operators $(a, a^\dagger)$ is derived. It corresponds to the unitary irreps (irreducible representations) of $SU(1,1)$ with a state of lowest weight which are characterized by a number $k > 0$. The $SU(1,1)$ coherent states to these irreps are discussed and it is shown that they are eigenstates of a non-Hermitean operator. For each $k > 0$, there exists a countable number of subdivisions of the Fock space spanned by the basis vectors $|sn + j\rangle$, $(n = 0, 1, \ldots)$ with fixed values $s = 1, 2, \ldots$ and $j = 0, \ldots, s-1$. The same is true for the realizations of the Heisenberg-Weyl algebra in the Fock space by basis operators $(A, A^\dagger, I = [A, A^\dagger])$. The coherent phase states are discussed as an example of $SU(1,1)$ coherent states. Some of their properties are related to the unorthodox integer function $f(z) = \sum_{n=0}^{\infty} z^n/\sqrt{n!}$ for which the first 4 pairs of its complex conjugated zeros are determined. The phase-optimized states are discussed and it is found that they hardly can be accepted as really "phase-optimized". The roots of the failure to find a Hermitean phase operator are found already in classical mechanics in a grave topological defect of the transition from canonical coordinates to action-angle coordinates as a canonical transformation in the coordinate origin.

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1 Introduction

Group theory is a powerful mean to join things which, on the first glance, seem to be completely disjointed. Group theory also provides means to make the disentanglement of complicated operations or to prove important relations. Group theory with its origins going back far to the 19-th century made its way into quantum mechanics from the first day of its existence on [1] but mainly into the theory of atoms, molecules and solids and of elementary particles. The

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infiltration of group theory into quantum optics came, apart from the use of the Heisenberg-Weyl group from its very beginning, mainly from the seventies on, with the group $SU(1,1)$ for squeezing, e.g., [2–25] and with the group $SU(2)$, for example, for two-mode systems (beamsplitter, light polarization) and with some other groups (e.g., higher symplectic groups $Sp(2n,R)$ for n-mode systems).

After construction of the abstract basis of the unitary irreps (irreducible representations) of $SU(1,1)$ in section 2, we consider in section 3 the $SU(1,1)$ coherent states. In section 4, we give the possible realizations of the Lie algebra $su(1,1)$ and of the Heisenberg-Weyl algebra by a pair of boson annihilation and creation operators. Squeezed vacuum states and coherent (or $SU(1,1)$) phase states, e.g., [6, 14, 16, 19, 23, 26] with the limiting case of London phase states [27] are examples of $SU(1,1)$ coherent states. We consider these phase states in sections 5 and 6 and discuss some of their properties and touch there the phase operator problem, e.g., [28–31] (and huge literature not cited here). In section 7, we discuss the "phase-optimized" states [32] which are very different from the coherent phase states and show that they hardly can be considered as genuinly "phase-optimized" due to the applied variance measure of phase. We also discuss shortly the "necessary" failure to find a Hermitean phase operator and related unitary operators corresponding to classical phase functions $\mathrm{e}^{in\varphi}$, $(n = \pm 1, \pm 2, \ldots)$. The roots of this failure are already to find in classical mechanics in the fact that the transition from canonical variables to action-angle variables possesses a grave topological defect as a canonical transformation in the coordinate origin [33].

2 Abstract Lie algebra $su(1,1)$ and basis of unitary representations

We now construct a basis to the unitary irreps of the abstract Lie algebra $su(1,1)$ [4]. The Lie algebra $su(1,1)$ is a 3-dimensional algebra and as a standard basis are mostly used 3 operators (K_0, K_1, K_2) or (K_0, K_-, K_+) which obey the following commutation relations

$$
\begin{aligned}
&[K_0, K_-] = -K_-, \qquad [K_0, K_+] = +K_+, \qquad [K_-, K_+] = 2K_0, \\
&K_\pm \equiv K_1 \pm \mathrm{i}K_2, \qquad C \equiv (K_0)^2 - \frac{1}{2}\left(K_-K_+ + K_+K_-\right),
\end{aligned} \tag{2.1}
$$

where C denotes the Casimir operator. A representation of a Lie algebra is a mapping of the elements x of the abstract Lie algebra into linear operators $T(x)$ of a Hilbert space with preservation of the structural relations, i.e., of the commutation relations and analogously for the abstract Lie group. For the unitary irreps of $su(1,1)$, one has to postulate the additional relations

$$
K_0 = K_0^\dagger, \qquad K_- = K_+^\dagger, \qquad K_+ = K_-^\dagger. \tag{2.2}
$$

The unitary irreps of $SU(1,1)$ are, with exception of the identity representation, infinite-dimensional representations. There exist finite-dimensional irreps of $su(1,1)$ to each given dimension which, however, do not lead to unitary representations of $SU(1,1)$.

We construct an orthonormalized basis to the unitary irreps of $su(1,1)$ by the eigenstates $|k,n\rangle$ of the Hermitean operator K_0 to eigenvalues $k+n$ with a lowest state $|k,0\rangle$ which is annihilated by the operator K_- that means

$$
K_0|k,n\rangle = (n+k)|k,n\rangle, \qquad K_-|k,0\rangle = 0. \tag{2.3}
$$

Then by using the Lie algebra $su(1,1)$ given by (2.1), one finds that K_+ is a raising operator and K_- a lowering operator of the eigenstates $|k,n\rangle$ of K_0 which raise or lower the eigenvalues in unit steps and in a straightforward way this leads to the relations

$$\begin{aligned} K_0|k,n\rangle &= (n+k)|k,n\rangle, \\ K_-|k,n\rangle &= \sqrt{n(n-1+2k)}\,|k,n-1\rangle, \\ K_+|k,n\rangle &= \sqrt{(n+1)(n+2k)}\,|k,n+1\rangle, \\ C|k,n\rangle &= k(k-1)|k,n\rangle, \qquad n=0,1,\dots\,, \qquad k>0. \end{aligned} \tag{2.4}$$

The positivity of the index k of the unitary irreps follows from $0 < \langle k,0|K_-K_+|k,0\rangle = 2k\langle k,1|k,1\rangle$. The basis states $|k,n\rangle$ are orthonormalized and complete within the given representation. This representation is constructed by Perelomov [4] (slightly different notations for the basis states). The derived set of unitary irreps is called in [4] the discrete series T^+ for values $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$. The extension to continuous $k > 0$ has a connection to the different global structure of $SU(1,1)$ and of its unitary irreps (related to universal covering group [4] and to metaplectic groups).

3 SU(1,1)-coherent states and negative binomial distributions

There are different definitions of generalized coherent states using generalizations of different aspects of the usual coherent states. One of the most effective ones is to generalize the group aspect of the usual coherent states that leads to group coherent states in the sense of Perelomov [4, 7, 8]. In application to $SU(1,1)$ this means that we look for the eigenstates $|k,0\rangle$ of the operator K_0 of lowest weight which are annihilated by the operator K_- and which play the role of the "vacuum state" of the constructed representation. Generalized group coherent states are then obtained by applying the whole set of group operators to arbitrary states of the irrep. Another possibility is to generalize the property of usual coherent states to be the eigenstates of the annihilation operator and that this operator annihilates the vacuum state. In application to $SU(1,1)$ this means that one looks for the eigenstates of the operator K_- in an irrep [9, 15, 19]. We consider only $SU(1,1)$ coherent states according to [4].

The action of the set of group elements of $SU(1,1)$ onto the state of lowest weight $|k,0\rangle$ is equivalent to the action of the nonunitary operator $\exp(zK_+)$ with complex parameter $z \in \mathrm{C}$ onto $|k,0\rangle$ ([4] and many others, e.g., [11, 14–16, 19]

$$|k,z\rangle \equiv N(k,|z|)\exp(zK_+)|k,0\rangle = (1-|z|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}\, z^n |k,n\rangle, \tag{3.1}$$

with $N(k,|z|)$ the normalization factor. It shows that the $SU(1,1)$ coherent states $|k,z\rangle$ are normalizable for complex z with $|z| < 1$ that means within the unit disk. This the nonunitary approach [24, 25] because $\exp(zK_+)$ is a nonunitary operator. If we denote with $S(\xi,\eta,\zeta)$ the following, in general, nonunitary $SL(2,C) \sim Sp(2,C)$ operator (complex extension of $SU(1,1)$) then one can derive different operator orderings by means of the fundamental two-dimensional representation of $SL(2,C)$, among them the following in analogy to normal ordering of boson operators

$$
\begin{aligned}
S(\xi,\eta,\zeta) &\equiv \exp\left(\xi K_- + \mathrm{i}2\eta K_0 - \zeta K_+\right) \\
&= \exp\Big(-\frac{\mu}{\kappa}K_+\Big)\exp\Big(-2\left(\log\kappa\right)K_0\Big)\exp\Big(\frac{\lambda}{\kappa}K_-\Big),
\end{aligned} \tag{3.2}
$$

with notations taken from the fundamental two-dimensional representation of $SL(2,C)$ by quadratic combinations of boson operators in the basis $(a, a^\dagger)$ [24]

$$
\begin{pmatrix} \kappa, & \lambda \\ \mu, & \nu \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\epsilon - \mathrm{i}\eta\dfrac{\mathrm{sh}\epsilon}{\epsilon}, & \xi\dfrac{\mathrm{sh}\epsilon}{\epsilon} \\ \zeta\dfrac{\mathrm{sh}\epsilon}{\epsilon}, & \mathrm{ch}\epsilon + \mathrm{i}\eta\dfrac{\mathrm{sh}\epsilon}{\epsilon} \end{pmatrix}, \qquad \epsilon \equiv \sqrt{\xi\zeta - \eta^2}. \tag{3.3}
$$

In the special case $\xi = -z^*, \eta = 0, \zeta = -z$ this leads to the identity

$$
\begin{aligned}
S\left(-\frac{z^*}{|z|}\mathrm{Arth}|z|, 0, -\frac{z}{|z|}\mathrm{Arth}|z|\right) &\equiv \exp\left(\frac{\mathrm{Arth}|z|}{|z|}(zK_+ - z^*K_-)\right) \\
&= \exp(zK_+)(1 - zz^*)^{K_0}\exp(-z^*K_-).
\end{aligned} \tag{3.4}
$$

This is the unitary approach to $SU(1,1)$ coherent states.

The $SU(1,1)$ coherent states $|k,z\rangle$ are eigenstates of the nonunitary operator $(K_0+kI)^{-1}K_-$ to complex eigenvalues z

$$
(K_0 + kI)^{-1}K_-|k,z\rangle = z|k,z\rangle. \tag{3.5}
$$

This can be proved by showing the equivalence of the action of the operator K_- to the action of the operators $z(K_0 + kI)$ onto the complete set of states $|k,z\rangle$ for arbitrary fixed z. We see here that $|k,z\rangle$ are not eigenstates of the operator K_- itself as in the other definition of $SU(1,1)$ coherent states [9] (or intelligent states [15, 19]). For example, vacuum states $|0\rangle$ squeezed are eigenstates of the operator $(N + I)^{-1}a^2$ and Fock states $|1\rangle$ squeezed of the operator $(N + 2I)^{-1}a^2$, whereas the eigenstates of a^2 lead to superpositions of even and odd coherent states [2, 10].

From (3.1), one finds the following distribution $p(k,n)$ for the diagonal elements of the $SU(1,1)$ coherent states with index k of the irrep (we write here the Gamma functions by corresponding factorials)

$$
p(k,n) \equiv \langle k,n|k,z\rangle\langle k,z|k,n\rangle = (1 - |z|^2)^{2k}\frac{(2k-1+n)!}{(2k-1)!n!}|z|^{2n}. \tag{3.6}
$$

This is a negative binomial distribution $p(n) = \binom{l-1+n}{l-1} p^l q^n$ with $l = 2k, q = |z|^2, p = 1 - q = 1 - |z|^2$. Thus the $SU(1,1)$ coherent states are the pure (or maximum coherent) states with negative binomial distributions of the diagonal elements $\langle k,n|k,z\rangle\langle k,z|k,n\rangle$ or their interpolation for noninteger $l = 2k$. The maximum incoherent states with binomial distribution are for $l = 2k = 1$ the thermal states. Many aspects of pure and mixed states with negative binomial dustribution are investigated in [22].

4 Realizations of $su(1,1)$ by boson operators

The space of true unitary irreps of the Heisenberg-Weyl group to a single mode is the Fock space which is spanned by the Fock states $|n\rangle$, $(n = 0, 1, 2, \ldots)$. We now consider the following realization of the Lie algebra $su(1,1)$ given by the action of the operators (K_0, K_-, K_+) onto the Fock states $|sn + j\rangle$

$$\begin{aligned}
&K_0|sn+j\rangle = (n+k)|sn+j\rangle, \\
&K_-|sn+j\rangle = \sqrt{n(n-1+2k)}\,|s(n-1)+j\rangle, \quad \Rightarrow \quad K_-|j\rangle = 0, \\
&K_+|sn+j\rangle = \sqrt{(n+1)(n+2k)}\,|s(n+1)+j\rangle, \\
&n = 0,1,2,\ldots, \qquad s = 1,2,\ldots\,, \qquad j = 0,1,\ldots,s-1.
\end{aligned} \tag{4.1}$$

If we identify the Fock states $|sn + j\rangle$ with the states $|k, n\rangle$ in the abstract representation of $su(1,1)$ in (2.4), then we have correspondence and (4.1) is a realization of the abstract Lie algebra $su(1,1)$. We have in these relations 3 parameters. The parameter $k > 0$ is the index of the irrep of $su(1,1)$, whereas $s = 1, 2, \ldots$ and $j = 0, 1, \ldots$ are two integer parameters which specialize the partial Fock space in which the irrep acts. One can express (K_0, K_-, K_+) given by the action onto Fock states in (4.1) in pure form by functions of the annihilation and creation operator $(a, a^\dagger)$ and of $N \equiv a^\dagger a$ in the following way (cf. other presentations in [13, 15])

$$\begin{aligned}
K_0 &= \frac{1}{s}\left\{N + (sk-j)I\right\}, \\
K_- &= \frac{1}{s}\sqrt{(N+(s-j)I)(N+(2sk-j)I)\frac{N!}{(N+sI)!}}\; a^s, \\
K_+ &= \frac{1}{s}a^{\dagger s}\sqrt{(N+(s-j)I)(N+(2sk-j)I)\frac{N!}{(N+sI)!}}.
\end{aligned} \tag{4.2}$$

We emphasize that these operators do not realize an irrep of $su(1,1)$ in the whole Fock space $\mathcal{H}$ but only in the partial Fock spaces $\mathcal{H}_j^{(s)}$ spanned by the corresponding states $|sn + j\rangle$ or two irreps with different indices k (e.g, usual squeezing $k = \frac{1}{4}, \frac{3}{4}$).

The Heisenberg-Weyl algebra $w(1, R)$ of a single boson mode admits also other realizations than the usual by the three basis operators $(a, a^\dagger, I)$ in the Fock space. Such realizations of the basis operators denoted by $(A, A^\dagger, I = [A, A^\dagger])$ are [13]

$$\begin{aligned}
&A|sn+j\rangle = \sqrt{n}\,|s(n-1)+j\rangle, \quad A^\dagger|sn+j\rangle = \sqrt{n+1}\,|s(n+1)+j\rangle, \\
&A^\dagger A|sn+j\rangle = n|sn+j\rangle, \quad AA^\dagger|sn+j\rangle = (n+1)|sn+j\rangle, \\
&(n = 0,1,\ldots), \qquad s = 1,2,\ldots, \qquad j = 0,1,\ldots,s-1,
\end{aligned} \tag{4.3}$$

or in pure operator form as operators of partial Fock spaces $\mathcal{H}_j^{(s)}$

$$\begin{aligned}
&A = \sqrt{\frac{N+(s-j)I}{s}\,\frac{N!}{(N+sI)!}}\; a^s, \quad A^\dagger = a^{\dagger s}\sqrt{\frac{N+(s-j)I}{s}\,\frac{N!}{(N+sI)!}}, \\
&A^\dagger A = \frac{1}{s}(N - jI), \qquad [A, A^\dagger] \equiv AA^\dagger - A^\dagger A = I, \qquad N \equiv a^\dagger a,
\end{aligned} \tag{4.4}$$

with I the unity operators of $\mathcal{H}_j^{(s)}$ to fixed (s,j) and with direct sum $\sum_{j=0}^{s-1} \oplus \mathcal{H}_j^{(s)} = \mathcal{H}$ for arbitrary s. The corresponding coherent states are obtained from the Fock-state expansion of the coherent states by the substitution $|n\rangle \to |sn+j\rangle$ and they are eigenstates of the operator A. For $s=1, j=0$, one obtains the well-known realization of the Heisenberg-Weyl algebra in the whole Fock space $\mathcal{H}$.

In the two-mode case, the Heisenberg-Weyl algebra $w(2,R)$ possesses 5 basis operators $(a_1, a_1^\dagger, a_2, a_2^\dagger, I)$. The outer automorphism group to the corresponding Heisenberg-Weyl group $W(2,R)$ which leaves invariant the commutation relations is the symplectic group $Sp(4,R)$ with 10 real parameters. It is isomorphic to the group $SO(3,2)$ [5] and contains many interesting subgroups, in particular, substantially different groups $SU(1,1)$ but also the group $SU(2)$ (lossless beamsplitter, two mode light polarization). The following two mode realization of the Lie algebra $su(1,1)$ is well known, e.g., [5, 14]

$$K_- = a_1 a_2, \quad K_0 = \frac{1}{4}\left(a_1 a_1^\dagger + a_1^\dagger a_1 + a_2 a_2^\dagger + a_2^\dagger a_2\right), \quad K_+ = a_1^\dagger a_2^\dagger. \tag{4.5}$$

As in the single mode case, there are many other possible realizations of $su(1,1)$ in the two-mode case by (fractal) functions of the annihilation and creation operators in generalization of (4.1) and (4.2), however, we do not discuss this here (cf. [13]).

5 Coherent phase states

Coherent phase states are $SU(1,1)$ coherent states in the realization of $su(1,1)$ by (4.2) with $k=\frac{1}{2}, s=1, j=0$ that, taking into account (3.8), corresponds to [14, 19]

$$\begin{aligned} K_- &= \sqrt{N+I}\,a, \qquad K_0 = \frac{1}{2}(aa^\dagger + a^\dagger a), \qquad K_+ = a^\dagger\sqrt{N+I}, \\ E_- &\equiv \left(K_0 + \frac{1}{2}I\right)^{-1} K_- = \frac{1}{\sqrt{N+I}}\,a, \qquad E_+ \equiv E_-^\dagger = a^\dagger \frac{1}{\sqrt{N+I}}. \end{aligned} \tag{5.1}$$

The commutation relations of powers of (E_-, E_+) with the number operator $N \equiv a^\dagger a$ are

$$[N, E_-^l] = -lE_-, \qquad [N, E_+^l] = +lE_+, \qquad l = 0, 1, \ldots\ . \tag{5.2}$$

The operators E_- and E_+ are nonunitary operators although the defect of the unitarity seems to be not "important". One can use E_- as one possible quantum-mechanical equivalent for the classical phase function $\mathrm{e}^{\mathrm{i}\varphi}$ and E_+ for $\mathrm{e}^{-\mathrm{i}\varphi}$. This is the Susskind-Glogower substitution [28,29] and, e.g., [16,26] (history, see [30]). The best substitution for the classical $\mathrm{e}^{\mathrm{i}\varphi}$, however, seems to be the symmetrically ordered function $\mathcal{S}\{a/\sqrt{aa^\dagger}\}$. Its use is related to the use of phase functions obtained by radius-integrated Wigner quasiprobabilities [31] but the results are mostly similar to the Susskind-Glogower substitution.

Already the transformation from canonical coordinates (q,p) to action-angle coordinates (j,φ) in classical mechanics possesses a grave topological defect as a canonical transformation at the coordinate origin $q=p=0$ (and only there) and, as a consequence, the quantum-mechanical transition from canonical operators (Q,P) to action-angle operators (J,Φ) cannot be a unitary transformation that prevents to obtain Φ as a Hermitean operator or $\exp(i\Phi)$ as a unitary operator [33].

For the coherent phase states, we introduce the special notation $|\varepsilon\rangle$. According to (3.1) and (3.2) they are determined in the nonunitary approach by ($s=1, j=0, k=\frac{1}{2}$)

$$|\varepsilon\rangle \equiv \sqrt{1-\varepsilon\varepsilon^*}\,\exp\left(\varepsilon a^\dagger\sqrt{N+I}\right)|0\rangle = \sqrt{1-\varepsilon\varepsilon^*}\sum_{n=0}^{\infty}\varepsilon^n|n\rangle. \tag{5.3}$$

In the unitary approach this is equivalent according to (3.7) to

$$|\varepsilon\rangle = \exp\left\{\frac{\mathrm{Arth}|\varepsilon|}{|\varepsilon|}\left(\varepsilon a^\dagger\sqrt{N+I} - \varepsilon^*\sqrt{N+I}\,a\right)\right\}|0\rangle. \tag{5.4}$$

The states $|\varepsilon\rangle$ are right-hand eigenstates of the operator $(K_0+\frac{1}{2}I)^{-1}K_-$ which is usually directly introduced and denoted by E_-

$$E_-|\varepsilon\rangle = \varepsilon|\varepsilon\rangle, \quad \Leftrightarrow \quad \langle\varepsilon|E_+ = \varepsilon^*\langle\varepsilon|. \tag{5.5}$$

The phase states become nonnormalizable for $|\varepsilon|\to 1$. By using $\varepsilon = |\varepsilon|\mathrm{e}^{\mathrm{i}\varphi}$ and by defining in the limiting case $|\varepsilon|\to 1$

$$|\varphi\rangle \equiv \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\mathrm{e}^{\mathrm{i}n\varphi}|n\rangle = \frac{1}{\sqrt{2\pi}}\exp\left(\mathrm{e}^{\mathrm{i}\varphi}a^\dagger\sqrt{N+I}\right)|0\rangle, \tag{5.6}$$

one obtains the London phase states [27] which are complete on the unit circle but nonorthogonal (discussion of complete relations in [23]).

Due to the Bunyakovski-Cauchy-Schwarz inequality, one has for states $|\psi\rangle$

$$V_0(|\psi\rangle\langle\psi|) \equiv \langle\psi|E_+E_-|\psi\rangle\langle\psi|\psi\rangle - \langle\psi|E_+|\psi\rangle\langle\psi|E_-|\psi\rangle \geq 0. \tag{5.7}$$

The coherent phase states $|\varepsilon\rangle$ as right-hand eigenstates of E_- and left-hand eigenstates of E_+ minimize this measure to $V_0(|\varepsilon\rangle\langle\varepsilon|)=0$. In [32] a similar measure

$$V_1(|\psi\rangle\langle\psi|) \equiv 1 - \langle\psi|E_+|\psi\rangle\langle\psi|E_-|\psi\rangle \geq \langle\psi|0\rangle\langle 0|\psi\rangle, \tag{5.8}$$

is used as a measure of the phase variance. It is obtained from (5.9) by substituting $E_+E_- \to E_-E_+ = I$ for normalizable states $|\psi\rangle$, $(\langle\psi|\psi\rangle = 1)$. Formally, if we make this substitution in (6.9), this measure would be minimized by the right-hand eigenstates of E_+ or left-hand eigenstates of E_-. However, these states are nonnormalizable and, therefore, the minimization problem under subsidiary conditions has to choose states among the normalizable states which become relatively complicated and were called "phase-optimized" states. They are far from the coherent phase states $|\varepsilon\rangle$. We consider these states in section 8.

The measures $V_0(|\psi\rangle\langle\psi|)$ and $V_1(|\psi\rangle\langle\psi|)$ provide essentially different values for states for which the vacuum state $|0\rangle$ is essentially occupied. This is the case for the coherent phase states $|\varepsilon\rangle$ for which one finds

$$V_0(|\varepsilon\rangle\langle\varepsilon|) = 0, \qquad V_1(|\varepsilon\rangle\langle\varepsilon|) = 1 - |\varepsilon|^2 = \frac{1}{\overline{N}+1}, \tag{5.9}$$

where $\overline{N} \equiv \langle\varepsilon|N|\varepsilon\rangle$ denotes the mean value of the number operator. Thus $V_1(|\psi\rangle\langle\psi|)$ is of the same order as for coherent states $|\alpha\rangle$ for which one obtains

$$V_0(|\alpha\rangle\langle\alpha|) \approx V_1(|\alpha\rangle\langle\alpha|) \approx \frac{1}{4|\alpha|^2} = \frac{1}{4\overline{N}}. \tag{5.10}$$

This is a true value for the phase variance of coherent states as can be affirmed by other ways of its estimation. However, it is inacceptable that for the coherent phase states $|\varepsilon\rangle$, one obtains by $V_1(|\varepsilon\rangle\langle\varepsilon|)$ for fixed $\overline{N}$ a value of the same order or larger as for coherent states $|\alpha\rangle$. To avoid misunderstanding, we emphasize that in many cases where the vacuum state is not essentially occupied the measures $V_1(|\psi\rangle\langle\psi|) \approx V_0(|\psi\rangle\langle\psi|)$ provide acceptable values for the phase variance but not for the coherent phase states themselves.

6 Some further relations for coherent phase states

The calculation of some properties of coherent phase states makes great difficulties since it is connected with an unorthodox integer function which is only given by its Taylor series expansion and which is difficult to investigate.

The mentioned difficulties can be illustrated by the calculation of quasiprobabilities and wave functions for coherent phase states. For the Husimi-Kano quasiprobability $Q(\alpha, \alpha^*)$ and for the Wigner quasiprobability $W(\alpha, \alpha^*)$ of coherent phase states, one has to calculate

$$
\begin{aligned}
Q(\alpha,\alpha^*) &= \frac{\langle\alpha|\varepsilon\rangle\langle\varepsilon|\alpha\rangle}{\pi} = \frac{1-\varepsilon\varepsilon^*}{\pi}\exp\left(-\alpha\alpha^*\right) f(\varepsilon^*\alpha) f(\varepsilon\alpha^*),\\
W(\alpha,\alpha^*) &= \exp\left(-\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) Q(\alpha,\alpha^*)\\
&= \frac{2(1-\varepsilon\varepsilon^*)}{\pi}\exp\left(-2\alpha\alpha^*\right)\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) f(2\varepsilon^*\alpha) f(2\varepsilon\alpha^*), \quad (6.1)
\end{aligned}
$$

and for the wave functions $\langle q|\varepsilon\rangle$ in position and $\langle p|\varepsilon\rangle$ in momentum representation

$$
\begin{aligned}
\langle q|\varepsilon\rangle &= \frac{\sqrt{1-\varepsilon\varepsilon^*}}{(\pi\hbar)^{\frac{1}{4}}}\exp\left(-\frac{q^2}{2\hbar}\right)\sum_{n=0}^{\infty}\frac{\varepsilon^n}{\sqrt{2^n n!}} H_n\left(\frac{q}{\sqrt{\hbar}}\right)\\
&= \frac{\sqrt{1-\varepsilon\varepsilon^*}}{(\pi\hbar)^{\frac{1}{4}}}\exp\left(-\frac{q^2}{2\hbar}\right)\exp\left(-\frac{\hbar}{4}\frac{\partial^2}{\partial q^2}\right) f\left(\sqrt{\frac{2}{\hbar}}\,\varepsilon q\right),\\
\langle p|\varepsilon\rangle &= \frac{\sqrt{1-\varepsilon\varepsilon^*}}{(\pi\hbar)^{\frac{1}{4}}}\exp\left(-\frac{p^2}{2\hbar}\right)\sum_{n=0}^{\infty}\frac{(-\mathrm{i}\varepsilon)^n}{\sqrt{2^n n!}} H_n\left(\frac{p}{\sqrt{\hbar}}\right)\\
&= \frac{\sqrt{1-\varepsilon\varepsilon^*}}{(\pi\hbar)^{\frac{1}{4}}}\exp\left(-\frac{p^2}{2\hbar}\right)\exp\left(-\frac{\hbar}{4}\frac{\partial^2}{\partial p^2}\right) f\left(-\mathrm{i}\sqrt{\frac{2}{\hbar}}\,\varepsilon p\right), \qquad (6.2)
\end{aligned}
$$

where $f(z)$ denotes the following integer (analytic) function of the variable $z = x + iy$

$$
f(z) \equiv \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}} = \pi^{\frac{1}{4}}\sum_{m=0}^{\infty}\frac{z^{2m}}{2^m m!}\left(\sqrt{\frac{m!}{(m-\frac{1}{2})!}} + \frac{z}{\sqrt{2}}\sqrt{\frac{m!}{(m+\frac{1}{2})!}}\right). \qquad (6.3)
$$

We see that the considered functions in (6.1) and (6.2) are connected by convolutions with the function $f(z)$ and all calculation difficulties go back to this unorthodox integer function $f(z)$ which is not considered in courses of mathematics.

In (6.3), we have separated the even and odd part of the function $f(z)$ and have applied the doubling formula for the Gamma function. The obtained representation shows that the function $f(z)$ grows for positive $z = x > 0$ faster than the Gaussian function $\exp(x^2/2)$. In the region of negative $z = x < 0$, there is a meticulous balance between the rapidly increasing even and odd parts of the function $f(x)$ which difference remains positive but decreases rapidly (not finally proved). It is difficult to prove or to develop exact relations for the function $f(z)$ such as, e.g., differential or integral equations, recursion relations, or argument transformations. It is even not simple to obtain graphical representations for a wide region of complex values z around the coordinate origin. For example, there appeared oscillating fields with angle at the wings to both sides of the essential regions of the Wigner quasiprobability and it was not clear, do they artificially arise from the abrupt truncation of the series or from the regions of negativities of the Wigner quasiprobability which have to be present since the Wigner quasiprobability for coherent phase states are not Gaussian functions.

Regions of negativities of the Wigner and other quasiprobabilities are coupled with the presence of zeros for some more smoothed quasiprobabilities, e.g., the Husimi-Kano quasiprobability $Q(\alpha, \alpha^*)$. According to (6.1) the zeros of $Q(\alpha, \alpha^*)$ are exactly determined by the zeros of the integer function $f(z)$ in the complex plane. We investigated all zeros of $f_N(z)$ which can be obtained by truncation of the Taylor series of $f(z)$ in (6.3) at values $n = N < 110$. From $N \approx 35$ on, it began to crystallize the first pair of complex conjugated zeros which remained extremely stable when going to higher approximations, in the region of about $N \approx 70$ the second pair and $N \approx 100$ the third pair. Once found, these zeros could be determined with high accuracy by going to higher approximations for this zero alone. However, in the region from $N > 90$ on, the calculations of all zeros with *Mathematica* began to become unreliable as checks have shown but, strangely, for some values of approximation in this region (e.g., $N = 90, 100, 108$), we got fully reliable results for all zeros of this approximation. We are almost sure but cannot prove this that in the region $|z| < 7$ there are no other zeros of $f(z)$ than the following 4 pairs of zeros

$$\begin{aligned} z_1^{\pm} &= 1.71697 \pm \mathrm{i}\, 3.18011, & z_2^{\pm} &= 2.88669 \pm \mathrm{i}\, 4.09032, \\ z_3^{\pm} &= 3.7571(6?) \pm \mathrm{i}\, 4.8189(3?), & z_4^{\pm} &= 4.47(866?) \pm \mathrm{i}\, 5.44(620?). \end{aligned} \tag{6.4}$$

The third pair of stable zeros could be determined yet with high accuracy but the fourth could be determined only with great trouble and is unreliable in some last figures and for the fifth, we did not obtain reliable values.

7 Phase-optimized or Lommel states

The "phase-optimized" states $|\psi\rangle$ were defined as the normalized states which minimize the phase variance measure $V_1(|\psi\rangle\langle\psi|) \equiv 1 - \langle\psi|E_-|\psi\rangle\langle\psi|E_+|\psi\rangle$ under the subsidiary conditions $\langle\psi|\psi\rangle = 1$ and $\langle\psi|N|\psi\rangle \equiv \overline{N}$ which means that $\overline{N}$ possesses an arbitrary fixed value [32]. The solution of this variation problem by the method with Lagrange multipiers and with the Ansatz of a Fock-state expansion $|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$ leads to the following recursion relation for the complex coefficients c_n

$$\mathrm{e}^{-\mathrm{i}\varphi} c_{n+1} - \alpha(\mu + n)c_n + \mathrm{e}^{\mathrm{i}\varphi} c_{n-1} = 0, \qquad c_{-1} = 0, \qquad (n = 0, 1, 2, \ldots). \tag{7.1}$$

This relation was solved with two real parameters α and μ (connected with $\overline{N}$ and the phase variance measure) by Bessel functions as follows [32]

$$|\varphi;\alpha,\mu\rangle^{(\mathrm{L})} = N(\alpha;\mu)\sum_{n=0}^{\infty} J_{\mu+n}\left(\frac{2}{\alpha}\right)\mathrm{e}^{\mathrm{i}n\varphi}|n\rangle, \qquad J_{\mu-1}\left(\frac{2}{\alpha}\right) = 0, \tag{7.2}$$

where the substitution of $J_\nu(z)$ by other cylindrical functions leads to nonnormalizable states. The condition $J_{\mu-1}(2/\alpha) = 0$ is necessary for $c_{-1} = 0$ that makes the recursion relations (7.1) consistent in the initial case $n = 0$. For given α it determines a series of solutions for μ from which the lowest has to be taken [32]. Other possible values lead only to larger relative mimima of the variation problem as the first although this seems to be not fully proved. The mentioned condition leads to a relation between the parameters α and μ with only discrete solutions and if we take the lowest solution for μ, we can eliminate the variable μ in the notation of the obtained states.

When solving the same recursion relations (7.1) in a successive way, we came in about 1992 to a solution by polynomials which proved to be the Lommel polynomials $R_{n,\mu}(z)$ [34, 35] as follows (presented in lectures and posters at conferencies, e.g., EQEC Firenze 1993 [33] but not published in a journal)

$$|\varphi;\alpha,\mu\rangle^{\mathrm{L}} = c_0\sum_{n=0}^{\infty} R_{n,\mu}\left(\frac{2}{\alpha}\right)\mathrm{e}^{in\varphi}|n\rangle,$$

$$R_{n,\mu}(z) \equiv \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!\Gamma(\mu+n-k)}{k!(n-2k)!\Gamma(\mu+k)}\left(\frac{2}{z}\right)^{n-2k}, \quad R_{0,\mu}(z) = 1. \tag{7.3}$$

The Lommel polynomials are related to Bessel functions by [34, 35]

$$J_{\mu+n}(z) = J_\mu(z)R_{n,\mu}(z) - J_{\mu-1}(z)R_{n-1,\mu+1}(z), \tag{7.4}$$

and for $J_{\mu-1}(z) = 0$ it follows

$$R_{n,\mu}(z) = \frac{J_{\mu+n}(z)}{J_\mu(z)}. \tag{7.5}$$

The use of the Lommel polynomials instead of the Bessel functions has some advantage but, nevertheless, for high values of z which are connected with high values of μ but moderate values $\overline{N}$, the calculations by computer become in both cases very cumberesome and one has to make calculations with high values of the indices $\mu + n$ (e.g. $2/\alpha = 1000, \mu = 982.454$ leading to $\overline{N} = 11.3575$). The Bessel functions can be substituted in a certain approximation by the Airy function $\mathrm{Ai}(z)$ due to a formula of Nicholson [32, 34] but the character of this approximation is not fully known [34] and should be checked by computer and the calculations with the Airy function also remain to be difficult. About the Lommel polynomials is only very little known and also this prevents to calculate many interesting characteristics of the states in an effective way. The "phase-optimized" states lower the occupation of the number states with low n from $n = 0$ on in comparison to the coherent phase states and, in the limiting case $\overline{N} \to \infty$, they do not make the transition to the London phase states (6.8).

Since the used measure for the phase dispersion in the determination of the discussed states provides an inacceptable large value for coherent phase states $|\varepsilon\rangle$ (section 6), it excludes these last states already at the beginning from the competition for phase optimization. The name "phase-optimized" states for the discussed states is confusing and it is better to call them "Lommel states" according to the involved polynomials.

8 Conclusion

After discussion of the realizations of $SU(1,1)$ by one-mode boson operators, there arises the analogous problem for the boson operators of two modes. Instead of the symplectic group $Sp(2,R) \sim SU(1,1)$, one has to consider the symplectic group $Sp(4,R)$ with 10 real parameters and with many interesting subgroups, among them different groups $SU(1,1)$ and $SU(2)$ and it arises the problem to find their possible realizations in the two-mode Fock space and to consider the corresponding group coherent states. By using realizations of the Heisenberg-Weyl algebra by differentiation and multiplication operators, the obtained realizations of $sp(2,R) \sim su(1,1)$ or, maybe, in future $sp(4,R)$ can be translated into groups of differentiation and multiplication operators that opens another wide field of applications. As an example, we considered the coherent phase states. Some of their characteristics as wave functions and quasiprobabilities are related to the unorthodox integer function $f(z) = \sum_{n=0}^{\infty} z^n/\sqrt{n!}$ which is difficult to deal with. The first 4 complex conjugated pairs of zeros of this functions were determined but more was not possible with our momentary means.

References

[1] H. Weyl: *Gruppentheorie und Quantenmechanik* (Hirzel, Leipzig, 1928)

[2] I.A. Malkin, V.I. Man'ko: *Dynamical Symmetry and Coherent States of Quantum Systems*, in Russian (Nauka, Moscow, 1979)

[3] J.R. Klauder, Bo-St. Skagerstam: *Coherent States* (World Scientific, Singapore, 1985)

[4] A.M. Perelomov: *Generalized Coherent States and Their Application* (Springer-Verlag, Berlin, 1986)

[5] Y.S. Kim, M.E. Noz: *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991)

[6] Fan Hongyi: *Representation and Transformation Theory in Quantum Mechanics*, in Chinese (Shanghai Scientific and Technical Publishers, Shanghai 1997)

[7] A.M. Perelomov: *Commun. Math. Phys.* **26** (1972) 222, republished in the collection of papers [3] on pp. 281–295.

[8] A.M. Perelomov: *Uspekhi Fiz. Nauk* **20** (1977) 703

[9] A.O. Barut, L. Girardello: *Commun. Math. Phys.* **21** (1971) 41, republished in the collection of papers [3] on pp. 255–269.

[10] V.V. Dodonov, I.A. Malkin, V.I. Man'ko: *Physica* **72** (1974) 597

[11] C.C. Gerry: *J. Phys. A: Math. Gen.* **16** (1983) L1

[12] J. Katriel, A.I. Solomon, G. D'Ariano, M.Rasetti: *Phys. Rev. D* **34** (1986) 2332

[13] G.M. D'Ariano, M.G. Rasetti, J. Katriel, A.I. Solomon: in *Squeezed States and Nonclassical Light*, edited by P. Tombesi and E.R. Pike (Plenum Press, New York, 1989) p. 301

[14] A. Vourdas: *Phys.Rev. A* **41** 3 (1990) 165
[15] V. Bužek: *J. Modern Opt.* **37** (1990) 303
[16] A. Vourdas: *Phys.Rev. A* **45** (1992) 1943
[17] A. Vourdas: *Phys.Rev. A* **46** (1992) 442
[18] A. Vourdas: *Physica Scripta* **48** (1993) 94
[19] C. Brif: *Quantum Semiclass. Opt.* **8** (1995) 803
[20] C. Brif: *Annals Phys.* (N.Y) **251** (1996) 180
[21] A. Vourdas: *Phys.Rev. A* **50** (1994) 3331
[22] A. Vourdas: *Phys.Rev. A* **51** (1995) 2353
[23] A. Vourdas, A. Wünsche: *J. Phys. A: Math. Gen.* **31** (1998) 9341
[24] A. Wünsche: *Ann. Phys.* (Leipzig) **1** (1992) 181
[25] A. Wünsche: in *Second International Workshop on Squeezed States and Uncertainty Relations*, Moscow 1992, NASA Conference Publication 3219 (Goddard Space Flight Center, Greenbelt, Maryland, 1993) p. 277
[26] A. Lukš, V. Peřinova: *Physica Scripta* **48** (1993) 94
[27] F. London: *Zeitschr. für Physik* **40** (1926) 193
[28] L. Susskind, J. Glogower: *Physics* **1** (1964) 49
[29] P. Carruthers, M. Nieto: *Rev. Mod. Phys.* **40** (1968) 411
[30] M.M. Nieto: *Physica Scripta* **48** (1993) 5
[31] B.M. Garraway, P.L. Knight: *Physica Scripta* **48** (1993) 66
[32] A. Bandilla, H. Paul, H.-H. Ritze: *Quantum Opt.* **3** (1991) 267
[33] A. Wünsche: in *Forth EQEC, Technical Digest*, Vol. 2 (Firenze, 1993) p.810
[34] H. Bateman, A. Erdélyi: *Higher Transcendental Functions*, Vol. II (McGraw–Hill Book, New York, 1953)
[35] G.N. Watson: *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1980)