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STOCHASTIC CONTROL OF QUANTUM DYNAMICS FOR A SINGLE TRAPPED SYSTEM *

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A stochastic control of the vibrational motion for a single trapped ion/atom is proposed. It is based on the possibility to continously monitor the motion through a light field meter. The output from the measurement process should be then used to modify the system's dynamics.

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1 Introduction

In recent years there has been an increasing interest on trapping phenomena and related cooling techniques [1]. A few years ago, it has been shown that a single ion/atom can be trapped and cooled down near to its zero-point vibrational energy state [2]. On the other hand, the continuous measurement of quantum systems is particularly relevant at this time because experimental technology is now at the point where *individual* quantum systems can be monitored continuously [3]. With these developments it should be possible in the near future to control quantum systems in real time by using the results of the measurement in the process of continuous feedback. The possibility to control trapped particles, indeed, gave rise to new models in quantum computation [4]. In these systems the two lowest vibrational states are used for logical operations among quantum bits and the dominant source of decoherence is the heating of the vibrational motion [5]. Here, a way to control the position of a trapped ion/atom is studied. It results the possibility of a motional phase space uncertainty contraction corresponding to localise the particle even below the quantum limit.

2 The Measurement Model

Let us consider one vibrational mode for a trapped system together with a meter mode. The Hamiltonian would be of the following form

$$H = H_m + \hbar\nu a^\dagger a + \hbar\chi\mathcal{O}X \,, \tag{1}$$

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where H_m is the free Hamiltonian of the meter, ν being the oscillation frequency of the ion/atom in the trap, and a, $a^\dagger$ the lowering and rising operators for the vibrational states of the trap. In the last term of Eq. (1), χ is the coupling constant, $X = (a + a^\dagger)/2$ is the dimensionless position operator, and $\mathcal{O}$ can be any meter operator obtained as linear combination of the H_m's eigenoperators $\mathcal{O}^\pm$ (but, for simplicity, in the following it is assumed as the sum of them). The practical implementation of such type of Hamiltonian will be discussed later on.

The evolution equation for the whole density operator D is assumed to be

$$\dot{D} = \mathcal{L}D - i\chi\,[\mathcal{O}X, D] + \frac{\kappa}{2}\left(2\mathcal{O}^- D\mathcal{O}^+ - \mathcal{O}^+\mathcal{O}^- D - D\mathcal{O}^+\mathcal{O}^-\right)\,, \tag{2}$$

where $\mathcal{L}$ describes the damped dynamics of the vibrational mode. Furthermore, the meter mode is considered heavily damped, so that the decay rate κ is very large, ($\kappa \gg \chi$), and the meter will almost always be in the lower eigenstate of H_m. This allows the adiabatic elimination of the meter mode by means of a perturbative calculation in the small parameter χ/κ, obtaining (see also Ref. [6])

$$D = [\rho \otimes \Pi] + \left(\frac{\chi}{\kappa}\right)\left[i\rho X \otimes \Pi\mathcal{O}^- + h.c.\right] + \left(\frac{\chi}{\kappa}\right)^2 [\ldots] + \ldots\,, \tag{3}$$

where $\rho = \mathrm{Tr}_m\, D$ is the reduced density matrix for the vibrational motion, and Π is the projector on the lower eigenstate of H_m.

Let us now suppose to be able to perform an homodyne-like measurement on the meter mode, i.e. to measure the observable $\mathcal{O}_\varphi = (\mathcal{O}^- e^{-i\varphi} + \mathcal{O}^+ e^{i\varphi})/2$. Then, the current at readout apparatus will be [7]

$$I(t) = 2\eta\kappa\langle\mathcal{O}_\varphi(t)\rangle_c + \sqrt{\eta\kappa}\,\xi(t)\,, \tag{4}$$

where η is the efficiency of the measurement process. The subscript c in Eq. (4) denotes the fact that the average is performed on the state conditioned on the results of the previous measurements and $\xi(t)$ is a Gaussian white noise [7]. Therefore, due to the interaction in Eq. (1), one gets information on X by observing the quantity $\mathcal{O}_\varphi$. From Eq. (3), the relationship between the conditioned mean values results $\langle\mathcal{O}_\varphi\rangle_c = (\chi/\kappa)\,\langle X\rangle_c\,\sin\varphi$.

However, the continuous monitoring of the meter mode modifies the time evolution of the whole system. In fact, the state conditioned on the result of measurement, described by the conditioned density matrix D_c, evolves according to the following stochastic differential equation (considered in the Ito sense)

$$\begin{aligned}\dot{D}_c &= \mathcal{L}D_c - i\chi\,[\mathcal{O}X, D_c] + \frac{\kappa}{2}\left(2\mathcal{O}^- D_c\mathcal{O}^+ - \mathcal{O}^+\mathcal{O}^- D_c - D_c\mathcal{O}^+\mathcal{O}^-\right)\\ &+ \sqrt{\eta\kappa}\,\xi(t)\left(e^{-i\varphi}\mathcal{O}^- D_c + e^{i\varphi} D_c\mathcal{O}^+ - 2\langle\mathcal{O}_\varphi\rangle_c\, D_c\right)\,.\end{aligned} \tag{5}$$

If we adopt the perturbative expression (3) for D_c in (5) and perform the trace over the meter mode, we get an equation for the reduced density matrix ρ_c conditioned to the result of the measurement of the observable $\mathcal{O}_\varphi$

$$\dot{\rho}_c = \mathcal{L}\rho_c - \frac{\chi^2}{2\kappa}\,[X, [X, \rho_c]] + \sqrt{\eta\chi^2/\kappa}\,\xi(t)\left(ie^{i\varphi}\rho_c X - ie^{-i\varphi}X\rho_c + 2\sin\varphi\langle X\rangle_c\,\rho_c\right)\,. \tag{6}$$

3 Applying the feedback loop

Let us now consider the application of a feedback loop to control the dynamics of the vibrational mode. The continous feedback theory proposed by Wiseman and Milburn [8] is then invoked. One has to take part of the stochastic output current $I(t)$, and feed it back to the vibrational dynamics in order to modify this. To be more specific, the presence of feedback modifies the evolution of the conditioned state $\rho_c(t)$. It is reasonable to assume that the feedback effect can be described by an additional term in the master equation, linear in the photocurrent $I(t)$, i.e. [8]

$$[\dot{\rho}_c(t)]_{fb} = \frac{I(t-\tau)}{\eta\chi}\,\mathcal{K}\rho_c(t)\,, \tag{7}$$

where τ is the time delay in the feedback loop and $\mathcal{K}$ is a Liouville superoperator describing the way in which the feedback signal acts on the system of interest.

The feedback term (7) has to be considered in the Stratonovich sense, since Eq. (7) is introduced as limit of a real process, then it should be transformed in the Ito sense and added to the evolution equation (6). A successive average over the white noise $\xi(t)$ yields, in the limiting case of $\tau \to 0$, the master equation

$$\dot{\rho} = \mathcal{L}\rho - \frac{\chi^2}{2\kappa}[X,[X,\rho]] + \mathcal{K}\left(ie^{i\varphi}\rho X - ie^{-i\varphi}X\rho\right) + \frac{\mathcal{K}^2}{2\eta\chi^2/\kappa}\rho. \tag{8}$$

The second term on the r.h.s. of Eq. (8) is the usual double-commutator term associated to the measurement of X, it results from the elimination of the meter variables; the third term is the feedback term itself and the fourth term is a diffusion-like term, which is an unavoidable consequence of the noise introduced by the feedback itself.

Due to the fact that the vibrational motion occurs at a frequency ν of the order of MHz and the damping rate of the center-of-mass motion, γ, is usually very small [9], it seems reasonable [10] to use the generator $\mathcal{L}$ as that of quantum optical master equation at a nonzero temperature [11] (indicating the number of thermal phonons with $n = [\exp(\hbar\nu/k_BT) - 1]^{-1}$). Moreover, since the Liouville superoperator $\mathcal{K}$ can only be of Hamiltonian form [8], it is taken as $\mathcal{K}\rho = g\left[a - a^\dagger, \rho\right]/2$, which means a driving term on the momentum, while g is the feedback gain related to the practical way of realizing the loop. Using the above expressions in Eq. (8) and rearranging the terms in an appropriate way, it is possible to get the following master equation:

$$\begin{aligned} \dot{\rho} &= \frac{\Gamma}{2}(N+1)\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right) + \frac{\Gamma}{2}N\left(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger\right) \\ &- \frac{\Gamma}{2}M\left(2a^\dagger\rho a^\dagger - a^{\dagger 2}\rho - \rho a^{\dagger 2}\right) - \frac{\Gamma}{2}M^*\left(2a\rho a - a^2\rho - \rho a^2\right) \\ &- \frac{g}{4}\sin\varphi\left[a^2 - a^{\dagger 2}, \rho\right] \end{aligned} \tag{9}$$

where $\Gamma = \gamma - g\sin\varphi$, and

$$N = \frac{1}{\Gamma}\left[\gamma n + \frac{\chi^2}{4\kappa} + \frac{g^2}{4\eta\chi^2/\kappa} + \frac{g}{2}\sin\varphi\right]\,;\; M = -\frac{1}{\Gamma}\left[\frac{\chi^2}{4\kappa} - \frac{g^2}{4\eta\chi^2/\kappa} - i\frac{g}{2}\cos\varphi\right]. \tag{10}$$

Eq. (9) clearly shows the effects of the feedback loop on the vibrational mode a. The proposed feedback mechanism, indeed, not only introduces a driving term, but it also simulates the presence of a bath with nonstandard fluctuations, characterized by an effective damping constant Γ and by the coefficients M and N, which are given in terms of the feedback parameters [6]. An interesting aspect of this effective bath is that it is characterized by phase-sensitive fluctuations.

Because of its linearity, the solution of Eq. (9) can be easily obtained in terms of the normally ordered characteristic function $\mathcal{C}(\lambda, \lambda^*, t)$ [11]. The stationary solution has the following form

$$\mathcal{C}(\lambda, \lambda^*, \infty) = \exp\left[-\zeta|\lambda|^2 + \frac{1}{2}\mu(\lambda^*)^2 + \frac{1}{2}\mu^*\lambda^2\right] , \tag{11}$$

where

$$\zeta = N + g\sin\varphi\frac{Ng\sin\varphi + \Gamma\mathrm{Re}\{M\} + (g\sin\varphi)/2}{\Gamma^2 - g^2\sin^2\varphi} ; \quad \mu = \frac{\Gamma}{g\sin\varphi}(\zeta - N) . \tag{12}$$

Under the stability conditions and in the long time limit ($t \to \infty$) the variance of the generic quadrature operator $X_\theta = (ae^{i\theta} + a^\dagger e^{-i\theta})/2$ becomes

$$4\langle X_\theta^2\rangle = 1 + 2\zeta + 2\mathrm{Re}\{\mu e^{2i\theta}\} . \tag{13}$$

For the position quadrature ($\theta = 0$), it can be simply written as

$$4\langle X^2\rangle = 1 + 2n_{eff} , \quad n_{eff} = \zeta + \mathrm{Re}\{\mu\} . \tag{14}$$

In absence of feddback ($g = 0$) we have $n_{eff} \equiv n$, otherwise n_{eff} can be smaller than n (the choice $\varphi = -\pi/2$ turns out to be the best), providing a *stochastic* localisation in the position quadrature, i.e. a *confinement*. Depending on the external parameters, it can also be negative (but it is always $n_{eff} \geq -1/2$) accounting for the possibility of going beyond the standard quantum limit. This is a relevant result of the present feedback scheme since it is able to reduce not only the thermal fluctuations but even the quantum ones.

4 The Practical Implementation

Let us now consider a trapped ion/atom in a Lamb-Dicke regime, in the presence of a standing wave field, and let us study different ways to implement the above model [12].

A Hamiltonian resembling Eq. (1) can be obtained by considering an atom located at the node and in resonant condition for which [13]

$$H = \hbar\Delta\sigma_z + \hbar\nu a^\dagger a + \hbar\chi\frac{(\sigma_+ + \sigma_-)}{2}X , \tag{15}$$

with σ_z, $\sigma_\pm$ the Pauli operators for two-level system, and Δ the-small-detuning between the atomic and the standing wave frequency. The coupling constant is given by the Rabi frequency times the Lamb-Dicke parameter. Hence, in this case, the electronic degree of freedom plays the role of meter, i.e. $\sigma_x \equiv \mathcal{O}$ and $\sigma_\pm \equiv \mathcal{O}^\pm$. By exploiting the resonance fluorescence it could be possible to get the quantity $\Sigma_\varphi = \left(\sigma_- e^{-i\varphi} + \sigma_+ e^{i\varphi}\right)/2$ through homodyne detection of the field scattered by the ion along a certain direction [14]. In fact, the detected field may be written in terms of the dipole moment operator for the transition $|-\rangle \leftrightarrow |+\rangle$ as $E_s^{(+)}(t) = \sqrt{\eta\kappa}\sigma_-(t)$ [11], where η is an overall quantum efficiency accounting for the detection efficiency and the fact that only a fraction of the fluorescent light is collected and superimposed with a mode-matched oscillator.

Alternatively, the off resonant situation can be also exploited. The starting Hamiltonian in this case implies the quantisation of the cavity field [11], and reads

$$H = \hbar\Delta\sigma_z + \hbar\nu a^\dagger a + \hbar\frac{2\epsilon^2}{\Delta}\sigma_z\, b^\dagger b \sin^2\left(\overline{k}X + \phi\right) \tag{16}$$

where b, $b^\dagger$ are the boson operators of the radiation mode, $\overline{k}$ is the dimensionless wave vector, ϵ is the dipole coupling constant, and Δ the-large-detuning.

By considering the internal atomic degree in the ground state and $\phi = \pi/4$, the leading term of Eq. (16) is

$$H = \hbar\nu a^\dagger a - \hbar\frac{2\epsilon^2}{\Delta}b^\dagger b\left(\overline{k}X + 1/2\right) . \tag{17}$$

After linearization around the steady state of the cavity mode, the dynamics will be governed by an effective Hamiltonian formally identical to that of Eq. (1)

$$H = \hbar\left(-\hbar\epsilon^2/\Delta\right)b^\dagger b + \hbar\nu a^\dagger a + \hbar\chi Y X \,, \tag{18}$$

where $Y = (b + b^\dagger)/2$, and $\chi = -4\beta\overline{k}\epsilon^2/\Delta$, with β the stationary value for the radiation amplitude (assumed real for semplicity). In this case the meter is represented by the cavity mode, i.e. $\mathcal{O} \equiv Y$, and $\mathcal{O}^+ \equiv a^\dagger$, $\mathcal{O}^- \equiv a$. The homodyne measurement of the light outgoing from the cavity allows to obtain the quantity $Y_\varphi = (ae^{-i\varphi} + a^\dagger e^{i\varphi})/2$ as desired.

Up to this point it has not considered how particular feedback Hamiltonians could be implemented. Of course it is important to be able to realize a term in the feedback Hamiltonian proportional to momentum. This is not so straightforward, but here possible ways in which it might be achieved are suggested. If the exact location of the trap is not an important consideration, then shifts in the position (being strictly equivalent to a linear momentum term in the Hamiltonian), are achieved simply by shifting all the position dependent terms in the Hamiltonian, in particular the trapping potential. This is a shift in the origin of the coordinates, and, being a virtual shift in the position, produces a term in the dynamical equation for the position proportional to the rate at which the trap is being shifted. When the experimental arrangement is such that the distance covered by the particle during the cooling is negligibly small compared to the trapping apparatus this may prove to be a very effective way of implementing a feedback Hamiltonian linear in momentum. Another method would be to apply a large impulse to the particle so that during one feedback time-step the particle is move the desired distance, and an equal and opposite impulse is then applied to reset the momentum. On the other hand, the use of laser pulses could be useful as well, since in accordance with the theory of laser cooling, the light exerts on the ion/atom a force proportional to its momentum.

5 Conclusion

Summarizing a stochastic control of the vibrational motion for a trapped particle via QND-mediated feedback has been proposed. In principle the model could be extended to the three dimensional case. The main limitation of such type of feedback is that it only acts on the measured (indirectly) variable, and it takes places only through a driving term in the variable conjugate to that measured.

However, there are many ways in which the measurement signal may be fed back to affect the system. In general, at a given time, any integral of the measurement record up until that time may be used to alter the system Hamiltonian and affect the dynamics. Hence the above schemes could be improved. To this end a promising technique seems to be the feedback via estimation [15].

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