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FROM THE CONTINUOUS MEASUREMENT THEORY BACK TO OPERATOR-VALUED PROCESSES *

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We show that a continuous-time Hermitian operator-valued process is measured in the continuous measurement. We illustrate the utility of the eigenkets of this quantum process for the explicit solution of the quantum stochastic equation describing the interaction between a field and a reservoir.

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1 Introduction

A Special Issue on Stochastic Quantum Optics [1] contains the paper by Barchielli and Paganoni [2], where both the theory of measurements continuous in time and the operator-valued processes are treated. The theory of continuous measurement was first formulated for the photon counter [3]. It was formulated so that it generalized the description of a damped harmonic oscillator coupled to a reservoir using master equation [4]. It enabled one to compare the detector with a zero-temperature reservoir. The continuous measurement under consideration is called a destructive measurement. It is very instructive, because the sum of the number of registrated counts (the conversion efficiency being unity) and the number of the "rest" photons is constant.

The theory of continuous measurement is utilized as a model of measurement of the photon number in a cavity mode using a sequence of atoms entering and exiting the cavity. It is proper when the sequence of atomic operators can be replaced by a continuous quantum process. To the experiments with atoms further continuous measurements can be applied, particularly, the nondemolition photon-number measurement. Here we learn of the photon number in a mode from the intensity of photocounting, which we must be able to estimate.

Lectures on the continuous measurement in quantum optics have been published by Carmichael in 1993 [5]. Of special interest is the limit of photon number measurement in the superposition of two modes, the second of which is in a strong coherent state. The limit procedure yields the description of quadrature measurement using the homodyne detection [6, 7]. The total number

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of registrated counts or photoelectrons, which is discrete, is replaced by a continuous integrated current (i.e., a charge) in the strong local oscillator limit. One continuous quantity, the time, therefore, is joined by a second one.

2 Continuous quantum measurement

Irreversibility may be incorporated in quantum theory by coupling a system to a zero-temperature uncorrelated (δ-correlated) reservoir and tracing over the reservoir. In case the system is a single-mode radiation field, this procedure provides a description of the system evolution by the master equation for the reduced statistical operator $\hat{\rho}(t)$,

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \frac{R}{2}\left(2\hat{O}\hat{\rho}\hat{O}^{\dagger} - \hat{O}^{\dagger}\hat{O}\hat{\rho} - \hat{\rho}\hat{O}^{\dagger}\hat{O}\right), \tag{1}$$

where $\hat{H}$ is a Hamiltonian, $\hat{O}$ is a Lindblad operator, and R is a kind of coupling constant of the field to the reservoir.

Recently, a number of theoretical methods have been developed which consider a vector-valued stochastic process, the vector being a pure state of a system [1,5]. Each of the trajectories of the process is conditioned on dissipation events at times $\tau_1, \tau_2, \ldots$. Typically, such events are connected to a continuous measurement. The system evolution is given up to a phase factor as nonunitary operators $\hat{u}_{[0,t)}(M, \tau_1, \ldots, \tau_M)$ comprising $\hat{O}$ as the operator which affects the state of the mode whenever a count is detected,

$$\begin{aligned} \hat{u}_{[0,t)}(M, \tau_1, \ldots, \tau_M) \;&=\; R^{\frac{M}{2}} \exp\left[-\frac{R}{2}(t-\tau_M)\hat{O}^{\dagger}\hat{O}\right] \\ &\times\; \hat{O}\exp\left[-\frac{R}{2}(\tau_M-\tau_{M-1})\hat{O}^{\dagger}\hat{O}\right]\ldots\hat{O}\exp\left(-\frac{R}{2}\tau_1\hat{O}^{\dagger}\hat{O}\right). \end{aligned} \tag{2}$$

The state of the radiation field conditioned on M dissipation events at times $\tau_1, \tau_2, \ldots, \tau_M$ is

$$|\psi_{|M,\tau_1,\ldots,\tau_M}(t)\rangle = \frac{|\psi_{M,\tau_1,\ldots,\tau_M}(t)\rangle}{\sqrt{P_{[0,t)}(M, \tau_1, \ldots, \tau_M)}}, \tag{3}$$

where

$$P_{[0,t)}(M, \tau_1, \ldots, \tau_M) = \langle\psi_{M,\tau_1,\ldots,\tau_M}(t)|\psi_{M,\tau_1,\ldots,\tau_M}(t)\rangle \tag{4}$$

and

$$|\psi_{M,\tau_1,\ldots,\tau_M}(t)\rangle = (-1)^M \hat{u}_{[0,t)}(M, \tau_1, \ldots, \tau_M)|\psi(0)\rangle. \tag{5}$$

In the cases $\hat{H} = \hat{0}$, $\hat{O} = \hat{a},\ \hat{a}^{\dagger},\ \hat{n}$, these operators have the properties

$$\hat{u}_{[0,t)}(M, \tau_1, \ldots, \tau_M) = \sqrt{P_{[0,t)}(\tau_1, \ldots, \tau_M|M)}\,\hat{u}_{[0,t)}(M), \tag{6}$$

where the factors are the square root of a conditional probability density and a simpler nonunitary operator, respectively. In the case of $\hat{O} = \hat{a}$, the conditional probability density and the simple nonunitary operator read

$$P^{(\mathrm{pc})}_{[0,t)}(\tau_1,\ldots,\tau_M|M) = \begin{cases} M!\left[\frac{R}{1-\exp(-Rt)}\right]^M \exp\left(-R\sum_{j=1}^M \tau_j\right) \\ \text{for } 0 \le \tau_1 < \ldots < \tau_M < t, \\ \\ 0 \ \text{otherwise}, \end{cases} \tag{7}$$

$$\hat{u}^{(\mathrm{pc})}_{[0,t)}(M) = \frac{[1-\exp(-Rt)]^{\frac{M}{2}}}{\sqrt{M!}} \exp\left(-\frac{Rt}{2}\hat{a}^\dagger\hat{a}\right)\hat{a}^M, \tag{8}$$

where the superscript (pc) stands for photon counter. In the case of $\hat{O} = \hat{a}^\dagger$, these quantities are of the form

$$P^{(\mathrm{qc})}_{[0,t)}(\tau_1,\ldots,\tau_M|M) = \begin{cases} M!\left[\frac{R}{\exp(Rt)-1}\right]^M \exp\left(R\sum_{j=1}^M \tau_j\right) \\ \text{for } 0 \le \tau_1 < \ldots < \tau_M < t, \\ \\ 0 \ \text{otherwise}, \end{cases} \tag{9}$$

$$\hat{u}^{(\mathrm{qc})}_{[0,t)}(M) = \frac{[\exp(Rt)-1]^{\frac{M}{2}}}{\sqrt{M!}} \exp\left(-\frac{Rt}{2}\hat{a}\hat{a}^\dagger\right)(\hat{a}^\dagger)^M, \tag{10}$$

where the superscript (qc) stands for quantum counter, and similarly in the case $\hat{O} = \hat{n}$ they are

$$P^{(\mathrm{nondem})}_{[0,t)}(\tau_1,\ldots,\tau_M|M) = \begin{cases} \frac{M!}{t^M} \ \text{for } \ 0 \le \tau_1 < \ldots < \tau_M < t, \\ \\ 0 \ \text{otherwise}, \end{cases} \tag{11}$$

$$\hat{u}^{(\mathrm{nondem})}_{[0,t)}(M) = \frac{(Rt)^{\frac{M}{2}}}{\sqrt{M!}} \exp\left(-\frac{Rt}{2}\hat{n}^2\right)\hat{n}^M, \tag{12}$$

where the superscript (nondem) refers to quantum nondemolition measurement.

In quantum optics it is useful to consider also the master equation (1), but with $\hat{O} = \hat{a}$ replaced by $\hat{O}_\xi = \hat{a} + \xi\hat{1}$ and $\hat{H} = \hat{0}$ by $\hat{H}_\xi = -i\hbar\frac{R}{2}(\xi^*\hat{a} - \xi\hat{a}^\dagger)$ [8]. In general, after such rearrangement of the master equation, the vector-valued stochastic process loses the property (6). When applied to the quadrature measurement with the aid of homodyne detection, ξ stands for the strength of the local oscillator.

3 Eigenkets of number operator-valued process

We will proceed with the measured system connected to a detector modelled as a quantum noise. We will work with the Hilbert space $\mathcal{H}_{\mathrm{e}} = \mathcal{H} \otimes \mathcal{H}_{[0,\infty)}$, where $\mathcal{H}$ is the single-mode separable Hilbert space and $\mathcal{H}_{[0,\infty)}$ is the 'continuous tensor product', i.e., the symmetrical Fock space of the quantum noise.

We choose the interaction Hamiltonian describing the interaction of the field mode and the detector

$$\hat{H}_{\rm int} = \hbar[\hat{G}^\dagger(t,t)\hat{O} + \text{H. c.}], \tag{13}$$

where H. c. means the Hermitian conjugate term to the previous one and $\hat{G}(s,t)$ for $s \geq 0$ is a quantum noise obeying the commutation relations

$$\begin{aligned} [\hat{G}(s,t), \hat{G}^\dagger(s',t)] &= R\delta(s-s')\hat{1}_{\rm e}, \\ [\hat{G}(s,t), \hat{G}(s',t)] &= \hat{0}_{\rm e}. \end{aligned} \tag{14}$$

We assume that the quantum process $\hat{G}(s,t)$, $s \geq 0$, is in the ground state initially and it is δ-correlated,

$$\begin{aligned} \langle\hat{G}(s,0)\rangle = \langle\hat{G}^\dagger(s,0)\rangle &= 0, \\ \langle\hat{G}^\dagger(s,0)\hat{G}(s',0)\rangle &= 0, \\ \langle\hat{G}(s,0)\hat{G}^\dagger(s',0)\rangle &= R\delta(s-s'), \\ \langle\hat{G}(s,0)\hat{G}(s',0)\rangle &= \langle\hat{G}^\dagger(s,0)\hat{G}^\dagger(s',0)\rangle = 0. \end{aligned} \tag{15}$$

We adopt the Schrödinger picture and we denote the Langevin force by

$$\hat{L}(t) = -i\hat{G}(t,0). \tag{16}$$

The mathematical theory of the stochastic differential equations, so useful for physics, does not in fact treat such differential equations, but the stochastic integral equations of the Volterra type. It is similar in the case of the quantum stochastic differential equations. The definition of the quantum stochastic integral is not elementary and we may distinguish the generalized Stratonovich and Itô approaches, respectively,

$$\int_0^t \{\mathcal{L}(t')|\psi_{\rm e}(t')\rangle\}_{\rm S}\,dt' = \lim_{\Delta\to 0}\sum_{j=0}^{J-1}\int_{t_j}^{t_{j+1}} \mathcal{L}(t')\,dt'\frac{1}{2}[|\psi_{\rm e}(t_j)\rangle + |\psi_{\rm e}(t_{j+1})\rangle], \tag{17}$$

$$\int_0^t \{\mathcal{L}(t')|\psi_{\rm e}(t')\rangle\}_{\rm I}\,dt' = \lim_{\Delta\to 0}\sum_{j=0}^{J-1}\int_{t_j}^{t_{j+1}} \mathcal{L}(t')\,dt'|\psi_{\rm e}(t_j)\rangle, \tag{18}$$

where $0 = t_0 < t_1 < \ldots < t_J = t$,

$$\Delta = \max_{0\leq j\leq J-1}(t_{j+1}-t_j), \tag{19}$$

$$\mathcal{L}(t) \equiv \hat{L}(t),\ \hat{L}^\dagger(t),\ \text{etc.} \tag{20}$$

Using the Stratonovich rule, we write the Schrödinger equation as if it were not δ-correlated,

$$\frac{d}{dt}|\psi_{\rm e}(t)\rangle = -\frac{i}{\hbar}\hat{H}|\psi_{\rm e}(t)\rangle + \{[\hat{L}(t)\hat{O}^\dagger - \hat{L}^\dagger(t)\hat{O}]|\psi_{\rm e}(t)\rangle\}_{\rm S}, \tag{21}$$

whereas using the Itô rule we rewrite it in the form

$$\frac{d}{dt}|\psi_{\rm e}(t)\rangle = -\frac{i}{\hbar}\hat{H}|\psi_{\rm e}(t)\rangle - \frac{R}{2}\hat{O}^\dagger\hat{O}|\psi_{\rm e}(t)\rangle + \{[\hat{L}(t)\hat{O}^\dagger - \hat{L}^\dagger(t)\hat{O}]|\psi_{\rm e}(t)\rangle\}_{\rm I}. \tag{22}$$

The solution of the two equivalent equations which obeys the initial condition

$$|\psi_{\rm e}(t)\rangle|_{t=0} = |\psi(0)\rangle \otimes |0\rangle_{[0,\infty)} \tag{23}$$

reads as

$$|\psi_{\rm e}(t)\rangle = \sum_{M=0}^{\infty}\int_0^t \dots \int_0^{\tau_2} |\psi_{M,\tau_1,\dots,\tau_M}(t)\rangle \otimes |M,\tau_1,\dots,\tau_M\rangle_{[0,\infty)} d\tau_1 \dots d\tau_M, \tag{24}$$

where $|\psi_{M,\tau_1,\dots,\tau_M}(t)\rangle$ is defined in (5). Here $|M,\tau_1,\dots,\tau_M\rangle_{[0,\infty)}$ are number states with the property

$${}_{[0,\infty)}\langle M,\tau_1,\dots,\tau_M|M',\tau_1',\dots,\tau_{M'}'\rangle_{[0,\infty)} = \delta_{MM'}\delta(\tau_1-\tau_1')\dots\delta(\tau_M-\tau_M'), \tag{25}$$

which are simultaneous eigenkets of the operators (cf. [9])

$$\frac{1}{R}\int_0^t \hat{L}^\dagger(t')\hat{L}(t')\,dt', \;\; 0 < t < \infty, \tag{26}$$

the eigenvalues $M(t)$ giving how many of numbers $\tau_1,\dots,\tau_M$ are less than t. The phase factor $(-1)^M$ in (5) is obtained, when it is supposed that

$$\hat{L}^\dagger(t)|M,\tau_1,\dots,\tau_M\rangle_{[0,\infty)} = \sqrt{R}\,|M+1,\tau_1,\dots,\tau_M,t\rangle_{[0,\infty)}. \tag{27}$$

In the cases where the relation (5) holds, "superposition" Fock states of the reservoir can be introduced and they suffice for the formal solution. Substituting from (5) and (6) into (24), we obtain the expansion

$$|\psi_{\rm e}(t)\rangle = \sum_{M=0}^{\infty} |\psi_M(t)\rangle \otimes |M\rangle_{[0,\infty)}(t), \tag{28}$$

where

$$|\psi_M(t)\rangle = (-1)^M \hat{u}_{[0,t)}(M)|\psi(0)\rangle \tag{29}$$

and

$$|M\rangle_{[0,\infty)}(t) = \int_0^t \dots \int_0^{\tau_2} \sqrt{P_{[0,t)}(\tau_1,\dots,\tau_M|M)}\;|M,\tau_1,\dots,\tau_M\rangle_{[0,\infty)}\,d\tau_1\dots d\tau_M. \tag{30}$$

The dynamics of the compound system described by the relation (24) is not that of the continuous measurement, because the measurement eigenkets, which resemble the quantum measurement so much, are in a coherent superposition here, whereas none of the trajectories is in any superposition in the continuous measurement. The individual measurement eigenkets are not very pleasant as they cannot be normalized, so that the uncertainties in the times of the detection events must be taken into account if the model of the continuous measurement should comprise individual normalized approximate measurement eigenkets.

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