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SYNTHESIS OF OPERATORS: UNIVERSAL QUANTUM GATES FOR A TRAPPED ION*

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We investigate physical implementations of universal quantum gates which perform *arbitrary* unitary transformations of *unknown* inputs. In particular, two approaches for synthesis of arbitrary unitary operators acting on vibrational states of a trapped ion are considered.

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1 Introduction

Remarkable progress in controlled manipulations of individual quantum systems (e.g., trapped ions and cold atoms in atomic physics and cavity QED) opens new perspectives for applications such as processing of quantum information and quantum computing which utilize fundamental principles of quantum physics [1]. There are several physical systems considered as candidates for implementation of quantum computing, for example, trapped ions [2] on which we focus our attention.

In general, processing of quantum information and quantum computing require to build from the available elementary systems quantum networks while keeping the coherent control over complex dynamics. Closely related tasks are preparation and manipulation of quantum states [1]. For example, *deterministic* quantum–state–synthesis procedures [3–5] offer an important tool to construct any desired state of quantum systems. However, a *complete control* of quantum systems requires engineering of arbitrary Hamiltonians which govern the unitary evolution [6, 7].

In this paper we consider realization of universal quantum gates for qubits which can be encoded in vibrational levels of a trapped ion. Our aim is to find a constructive algorithm to realize an arbitrary unitary operator $\hat{V}$ which transforms any unknown vibrational state $|\psi\rangle$ of a trapped ion to $|\psi'\rangle = \hat{V}|\psi\rangle$. We assume that quantum information is stored in a finite number

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of Fock states, i.e., $\hat{V}$ is defined only on a finite–dimensional Hilbert subspace $\mathcal{H}_N$ spanned by vibrational number states $\{|0\rangle, , \dots |N\rangle\}$.

We consider two approaches for synthesis of desired quantum gates. The first one is based on switching of two interaction channels in the spirit of Harel and Akulin [7]. The second one uses transparent encoding of the desired transformation matrix into number–state "lattice" of two vibrational modes (with one of the modes acting as an ancilla).

2 Engineering of unitary evolution

Recently Harel and Akulin have shown that by switching between two distant constant *perturbations* any desired unitary evolution of quantum systems can be attained. The power of the method has been illustrated in controlling the 1-D translational motion of a cold atom. The idea behind this method is quite general [6]. Specifically, as discussed by Reck et al. [7] any discrete finite–dimensional unitary matrix can be decomposed (factorized) into a sequence of 2×2 matrices corresponding to beam splitter transformations. Obviously, also other decompositions are possible. We assume that the unitary operator $\hat{V}$ can be written as an alternating sequence of switching between two evolution operators $\hat{U}^{(p)} = \exp[-iH^{(p)}\tau]$

$$\hat{V} = \hat{U}^{(b)}_{\xi_{M^2}} \hat{U}^{(a)}_{\xi_{M^2-1}} \dots \hat{U}^{(b)}_{\xi_2} \hat{U}^{(a)}_{\xi_1} \tag{1}$$

associated with two different interaction channels. To synthesize $M \times M$ unitary matrix we need M^2 *independent* real interaction parameters $\{\xi_j\}_{j=1}^{M^2}$. The requirement suggested in Refs. [6,7] to ensure the decomposition (1) is that the set of commutators

$$[\hat{H}^{(a)}, [\hat{H}^{(a)}, \dots [\hat{H}^{(a)}, \hat{H}^{(b)}], \quad [\hat{H}^{(b)}, [\hat{H}^{(b)}, \dots [\hat{H}^{(b)}, \hat{H}^{(a)}] \tag{2}$$

of all orders up to M^2 form a complete basis for the $M \times M$ unitary matrices.

The sequence (1) represents the system of nonlinear equations for M^2 real interaction parameters ξ_j. For properly chosen interaction channels the task requires a technique which ensures the solution (however not unique, in general). The novel method proposed by Harel and Akulin [7] is based on an "identity map" (with just M parameters) from which the required parameters ξ_j are found as "perturbations" which are solutions of a set of linear equations.

In our particular case we consider an implementation of universal quantum gates for a trapped ion. The motional degrees of freedom can be well controlled via irradiating the ion by two properly aligned laser beams (e.g. along x axis) which stimulate a Raman transition. Frequency difference of the "Raman" lasers can be tuned to the electronic transition or appropriate vibrational sidebands. As alternating "perturbations" we utilize the effective interaction Hamiltonians which are typical for recent experiments [2]:

$$\begin{aligned} \hat{H}^{(a)} &= g_a \mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)|b\rangle\langle c| + g_a^\star \mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)|c\rangle\langle b|, \\ \hat{H}^{(b)} &= g_b \hat{a}_x^\dagger \mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)|b\rangle\langle c| + g_b^\star \mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)\hat{a}_x|c\rangle\langle b|. \end{aligned} \tag{3}$$

Here $\hat{a}_x$, $\hat{a}_x^\dagger$ are the annihilation and creation operators for the vibrational mode x (i.e., two counter–propagating laser beams are aligned along the x axis), $|b\rangle, |c\rangle$ are the internal electronic levels. The nonlinear term $\mathcal{F}(\hat{a}_x \hat{a}_x) = \mathrm{e}^{-\eta_x^2/2} \sum_{k=0} \frac{(-1)^k \eta_x^{2k}}{(k+1)!k!} \hat{a}_x^{\dagger k} \hat{a}_x^k$ applies for operating the trap outside of the Lamb–Dicke regime; the Lamb–Dicke parameter is specified as

$\eta_x = \omega_x/(c\sqrt{2m_a\nu_x})$ (in units with $\hbar = 1$); ω_x, ν_x are frequencies of the laser field and vibrational mode in x direction; m_a is the mass of the ion. For more details see Refs. [2, 3]. Note that nonvanishing commutators (2) require to operate outside of the Lamb–Dicke regime with the nonlinear term $\mathcal{F}$. The independent interaction parameters are given as $\xi_j = |g_p|\mathrm{e}^{-i\varphi_j}\tau_j$ with $j = 1, \ldots, 2N^2$. It means that $2N^2$ real parameters represent interaction "lengths" $|g_p|\tau_j$ and the additional set of $2N^2$ real parameters are phases $\{\varphi_j\}$ of the interaction constants g_p of the interaction Hamiltonians $\hat{H}^{(p)}$ (3) which can be properly adjusted.[2]

It is worth noticing that the interaction Hamiltonians (3) cause transfer of amplitudes also outside the subspace $\mathcal{H}_N$ to higher number states which can be temporally occupied during intermediate steps in the sequence (1). Moreover, the practical realization of the decomposition (1) requires a truncation of the Fock space and matrices of unitary operators $\hat{U}^{(p)}$. Therefore, to realize the quantum gate $\hat{V}$ on $\mathcal{H}_N$ (just for the first $N+1$ number states) we extend the definition of the matrix V from $\mathcal{H}_N$ to $\mathcal{H}_{N'}$. For our particular choice $N' = 2(N+1)$ the extra diagonal elements are equal to unity and the others vanish. It means that instead of $\hat{V}$ its extension $\hat{V}'$

$$\hat{V}' = \begin{pmatrix} \hat{V} & 0 \\ 0 & \hat{1} \end{pmatrix} \tag{4}$$

is decomposed according to (1) with $M = 2N'$. In this way we can reduce the error due to the finite-dimensional truncation of the unitary operators involved.

Let us note that the transfer of amplitudes which leads to temporal occupations of higher Fock states [during intermediate steps in (1)] invokes questions about the influence of decoherence on higher number states as well as validity of the Hamiltonians (3) which assume the rotating wave approximation at trap frequencies.

To avoid even temporal excitations of the number states outside of the desired finite-dimensional subspace we can consider also different encoding of qubits. Besides of the vibrational mode x we can utilize also vibrational mode y. In particular, we consider subspace with a *fixed* total number of vibrational quanta $N-1$. As one of the "perturbations" we can again use $\hat{H}^{(a)}$ but instead of the one–mode Hamiltonian $\hat{H}^{(b)}$ in (3) we use the effective two–mode interaction Hamiltonian [8] which has been used in an algorithm for synthesis of 2D vibrational states [5]:

$$\hat{H}'^{(b)} = g'_b \hat{a}_x^\dagger \mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)\mathcal{F}(\hat{a}_y^\dagger \hat{a}_y)\hat{a}_y|b\rangle\langle c| + g'^{\star}_b \hat{a}_y^\dagger \mathcal{F}(\hat{a}_y^\dagger \hat{a}_y)\mathcal{F}(\hat{a}_x^\dagger \hat{a}_x)\hat{a}_x|c\rangle\langle b|. \tag{5}$$

Note that both Hamiltonians $\hat{H}^{(a)}$ and $\hat{H}'^{(b)}$ commute with the operator of the total number of vibrational quanta $\hat{N} = \hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y$. It means that by the sequence of switchings between these two interaction channels we can represent an arbitrary unitary quantum gate (operator) which acts on the *finite* dimensional subspace spanned by $N+1$ two–mode number states $\{|N,0\rangle, \ldots |0,N\rangle\}$. This approach completely avoids temporal occupations of the states outside of the original Hilbert subspace.

In the next section we discuss briefly another approach based on a direct encoding of the desired operator $\hat{V}$ into the "lattice" of two–mode number states as introduced in [5].

[2] The interaction constants $g_p \sim \mathcal{E}_a^*\mathcal{E}_b/\Delta$ where $\mathcal{E}_p$ are complex laser amplitudes and Δ is a detuning of the lasers from the off-resonant level of the Raman transition. In other words, the phases φ_j can be adjusted as a phase difference of the laser beams. On the other hand, if only interaction times $\{\tau_j\}$ are considered as adjustable parameters the sequence (1) has to be doubled up to $M = 2N$.

3 Constructive algorithm for synthesis of operators

Another kind of the implementation of the unitary transformation $\hat{V}$ is based on the mapping $|\psi,0\rangle \xrightarrow{\hat{V}} |0,\psi'\rangle = |0,\hat{V}\psi\rangle$ of two–mode vibrational states. In particular, we consider the ion to be confined in 2D trapping potential and we utilize the vibrational mode y as *ancilla*. Schematically we can write the realization of the transformation $\hat{V}$ as the mapping

$$|0,\psi'\rangle \otimes |b\rangle = (N+1)^{-1/2}\hat{P}_{|b\rangle}\,\hat{B}\,\hat{A}\,|\psi,0\rangle \otimes |a\rangle \tag{6}$$

where $|a\rangle$ is the third (auxiliary) electronic level. The operators $\hat{A}$ and $\hat{B}$ represent a sequence of four types of unitary operations (see below) and the final projection $\hat{P}_{|b\rangle}$ on the state $|b\rangle$ selects *conditionally* the right outcome.

Besides the interactions (3) we utilize two additional interaction channels associated with laser stimulated interactions. Namely, the following (effective) interaction Hamiltonians:

$$\begin{aligned}
&\hat{H}^{(1,m)} = (\Delta_y + \hat{s}_x)(|a\rangle\langle a| - |b\rangle\langle b|) + g_1|a\rangle\langle b| + g_1^\star|b\rangle\langle a|, \\
&\hat{H}^{(2)} = g_2\hat{a}_y^\dagger\mathcal{F}(\hat{a}_y^\dagger\hat{a}_y)|b\rangle\langle c| + g_2^\star\hat{a}_y\mathcal{F}(\hat{a}_y^\dagger\hat{a}_y)|c\rangle\langle b|, \quad \hat{H}^{(3)} = \hat{H}^{(a)}, \quad \hat{H}^{(4)} = \hat{H}^{(b)}
\end{aligned} \tag{7}$$

are considered. The dynamical Stark shift described by the operator $\hat{s}_x = \sum_m s_m|m\rangle_x\langle m|$ with $s_m = \chi\left[1 + \mathrm{e}^{-2\eta_x^2}L_m^0(4\eta_x^2)\right]$ is induced by a detuned standing–wave laser field applied in x direction [4]. For $\chi \gg |g_1|$ and the detuning of the laser field in the y direction $\Delta_y = -s_m$ the effective Hamiltonian $H^{(1,m)}$ resonantly addresses only the states with fixed number m of excitations in the mode x [4].

Step A. To be more explicit, the complex operator $\hat{A}$ in (6) denotes "writting" of the matrix $V_{n,m}$ (multiplied with the amplitudes of unknown state $|\psi\rangle = \sum_m c_m|m\rangle$) into the 2D vibrational "lattice" of number states $|m,n\rangle$. This writing is done "column by column" for the subspaces (columns) with a fixed number m of phonons in the x direction [particular columns are selected by means of $H^{(1,m)}$]. After the action of $\hat{A}$ the entangled state of the whole system becomes

$$|\psi^A\rangle = \sum_{m,n=0}^{N,N} V_{n,m}c_m\mathrm{e}^{i\varphi_m^A}|m,n\rangle \otimes |a\rangle. \tag{8}$$

The operator $\hat{A}$ can be written as

$$\begin{aligned}
&\hat{A} = \textstyle\prod_{m=0}^{N}\hat{U}^{(1,m)}_{-\pi/2\exp(i\phi_m)}\hat{A}(m)\hat{U}^{(1,m)}_{\pi/2}, \\
&\hat{A}(m) = \hat{U}^{(2)}_{|m,N;b\rangle}\hat{U}^{(3)}_{|m,N-1;c\rangle}\cdots\hat{U}^{(2)}_{|m,1;b\rangle}\hat{U}^{(3)}_{|m,0;c\rangle}.
\end{aligned} \tag{9}$$

The subscript of the unitary transformation $\hat{U}^{(1,m)}_{g_1\tau}$ indicates the required setting of the corresponding interaction parameter $g_1\tau = |g_1\tau|\mathrm{e}^{i\phi_m}$ (for clarity, all explicit expressions for parameters are given in the Lamb–Dicke limit $\eta_{x(y)} \to 0$). For other transformations $\hat{U}^{(p)}_{|\Phi\rangle}$ $(p = 2,3)$ the subscripts denote steps in 1D quantum-state synthesis as explained below.

According to Eq.(9), we can select the column with m phonons in x direction by means of $\hat{U}^{(1,m)}_{\pi/2}$ which flips the electronic level $|a\rangle$ to $|b\rangle$. On the selected column we apply 1-D quantum state synthesis as the action of the operator $\hat{A}(m)$ which "prepares" the superposition $\sum_{n=0}^{N} V_{n,m}|m,n\rangle\otimes|b\rangle$. After that the level $|b\rangle$ is flipped back to $|a\rangle$ as we apply $\hat{U}^{(1,m)}_{-\pi/2\exp(i\phi_m)}$ (choosing also the proper phase factors ϕ_m [5]).

The appropriate interaction parameters for the involved 1-D quantum state synthesis can be found when we solve the inverse task based on "sweeping" down the probability from the component states of the given superposition in the y mode into the vacuum: $|m,0\rangle\otimes|b\rangle = \hat{A}^\dagger(m)\sum_n V_{n,m}|m,n\rangle\otimes|b\rangle$. Thus the subscripts of the unitary operators $\hat{U}^{(p)}_{|\Phi\rangle}$ $(p=2,3)$ in (9) indicate that the interaction parameters $g_p\tau = e^{i\phi}|g_p\tau|$ are chosen in such way that after the action of the $\hat{U}^{\dagger(p)}_{|\Phi\rangle}$ the amplitude (population) of the component state $|\Phi\rangle$ becomes equal to zero (see Ref. [3] for details).

Note that we apply the state–synthesis procedure only in y direction and the resulting state (8) remains *unknown*. After the action of $\hat{A}$ the matrix elements $V_{n,m}$ are encoded into columns of the 2D vibrational "lattice" of number states $|m,n\rangle$. In the next step an appropriate superposition within "rows" of the 2D vibrational number–state "lattice" is performed.

Step B. The operator $\hat{B}$ makes the proper superpositions of the amplitudes within the particular rows (subspaces with fixed number n of phonons in y direction) of the vibrational "lattice". The resulting amplitudes of the states $|0,n\rangle\otimes|b\rangle$ $(n=0,...N)$ are proportional to $c'_n = \sum_m V_{n,m}\,c_m$. The state of the system $|\psi^B\rangle = \hat{B}|\psi^A\rangle$ after this synthesis step reads

$$|\psi^B\rangle = \frac{1}{\sqrt{N+1}}|0,\psi'\rangle\otimes|b\rangle + \sum_{m,n=0}^{N-1} z_{m,n}|m,n\rangle\otimes|a\rangle. \tag{10}$$

The operator $\hat{B}$ can be written in the form

$$\begin{aligned}\hat{B} &= \left[\prod_{m=0}^{N-1}\hat{B}(m)\right]\hat{U}^{(1,N)}_{-i\pi/2},\\ \hat{B}(m) &= \hat{U}^{(1,m)}_{-i\arctan(1/\sqrt{N+1-m})}\hat{U}^{(3)}_{-i\pi/2}\hat{U}^{(4)}_{-i\pi/(2\sqrt{m+1})}.\end{aligned} \tag{11}$$

The operator $\hat{B}$ acts in "parallel" on the rows with different n. First, the operator $\hat{U}^{(1,N)}_{-i\pi/2}$ transfers the population from the internal level $|a\rangle$ to $|b\rangle$ only on the column with $m=N$ phonons in the mode x. Further, this population is transfered to the level $|b\rangle$ on the neighboring column with $m=N-1$ performing the transformations $\hat{U}^{(3)}_{-i\pi/2}\hat{U}^{(4)}_{-i\pi/(2\sqrt{m+1})}$. The equal superposition of the amplitudes $V_{n,m}c_m$ and $V_{n,m-1}c_{m-1}$ for $m=N$ is obtained after action of $\hat{U}^{(1,m-1)}_{-i\arctan(1/\sqrt{2})}$, i.e. undergoing one half of the Rabi flipping. Decreasing m (the number of quanta in the mode x) the basic sequence $\hat{B}(m)$ is recurrently repeated. Note that $\hat{U}^{(1,m)}_{-i\arctan(1/\sqrt{N-m+1})}$ adds the amplitude $V_{n,m}c_m$ (with a proper weight) to previously superposed amplitudes $\sum_{k=m+1}^{N} V_{n,k}c_k$ while performing an adequate part of the Rabi flipping.

Step C. Comparing the state $|\psi^B\rangle$ (10) with the desired one we can see that the target state is entangled to the internal level $|b\rangle$ and undesired component states $|m\neq 0,n\rangle$ to $|a\rangle$. Therefore we can perform a conditional measurement to project the state vector (10) on the internal level $|b\rangle$.[3]

[3]The internal state can be determined by driving transition from $|a\rangle$ to an auxiliary level $|r\rangle$ and observing the fluorescence signal.

In spite of the conditional selection this algorithm is *universal*. Both the sequence of the "elementary" operations which represents the desired transformation (choosing appropriate interaction parameters) and the probability of the right outcome are *independent* of input (unknown) states.

4 Conclusions

One of the important applications of the synthesis of operators is a realization of *universal* quantum gates for qubits which can be encoded into vibrational levels of a trapped ion.

We have discussed two different approaches for synthesis of unitary operators for a trapped ion. The first one is based on a sequential switching of two different interactions. It was shown that to avoid temporal occupations of the states outside of the "target" Fock subspace a two–mode arrangement has to be required. The second approach is more transparent. This constructive algorithm involves a direct encoding of transformation matrices into the "lattice" of two–mode numbers states. However, a conditional selection of the right output is required. Both approaches are applicable outside of the Lamb–Dicke regime with the number of elementary operations which scales as N^2.

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