**acta physica slovaca vol. 49 No. 4, 567 – 578 August 1999**

# THE ISSUE OF PHASES IN QUANTUM MEASUREMENT THEORY*

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Received 21 May 1999, accepted 24 May 1999

The issue of phases is always very subtle in quantum world and many of the curious phenomena are due to the existence of the phase of the quantum mechanical wave function. We investigate the issue of phases in quantum measurement theory and predict a new effect of fundamental importance. We call a quantum system under goes a quantum Zeno dynamics (QZD) when the unitary evolution of a quantum system is interrupted by a sequence of measurements. In particular, We investigate the effect of repeated measurements on the geometric phase and show that the quantum Zeno dynamics can inhibit its development under a large number of measurement pulses. It is interesting to see that neither the total phase nor the dynamical phase goes to zero under large number of measurements. This new effect we call as the *"quantum Zeno Phase effect" (QZPE)* in analogous to the quantum Zeno effect (QZE) where the repeated measurements inhibit the transition probability. This "quantum Zeno Phase effect" can be proved within von Neumann's collapse mechanism as well as using a continuous measurement model. So the effect is really independent of any particluar measurement model considered. Since the geometric phase attributes a memory to a quantum system our results also proves that the path dependent memory of a system can be erased by a sequence of measurements. The QZPE provides a way to control and manipulate the phase of a wave function in an interference set up. Finally, we stress that the quantum Zeno Phase effect can be tested using neutron, photon and atom interference experiments with the presently available technology.

PACS: 03.65.Bz

## 1 Introduction

A theory with mathematical formalism of strong foundation, yet incapable of yielding a unique interpretation is the quantum theory of the present century. At the heart of the quantum theory lies the principle of linear superposition and the postulate of wavefunction collapse, which have given rise to several paradoxical problems over the years. We are confronted with two types

* Presented at the 6th Central-European Workshop on Quantum Optics, Château Chudobín, Czech Republic, April 30 - May 3, 1999.
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of basic dynamical evolutions in quantum theory, namely, one is the unitary, reversible evolution leading to deterministic solutions and other is non-unitary, irreversible evolution leading to probabilistic predictions. The former is referred to as Schrödinger evolution and the latter corresponds to von Neumann's collapse mechanism in the standard "Copenhagen interpretation" of quantum theory. These two types of dynamical evolution cannot be reconciled. The combination of unitary and reduction mechanism on a quantum system give some interesting effects. The unitary evolution creates a linear superposition of all possible eigenstates and measurement leads to the collapse of the linear superposition to a particular eigenstate corresponding to the eigenvalue of the observable being measured. On the other hand, if the unitary evolution is interrupted successively at short time intervals by measurements (we call such a dynamics as quantum Zeno dynamics (QZD)) the quantum system continues to remain in its initial state. This inhibition of quantum transition under frequent observations in rapid succession is called quantum Zeno effect (QZE). Misra and Sudarshan [1] were the first to predict the QZE for an unstable quantum system resulting in the inhibition of decay. There were various proposals to test the prediction of QZE. Due to a recent proposal of Cook [2], Itano *et al.* [3] have verified the inhibition of a quantum transition in $Be^+$ ions which are laser-cooled and trapped in a Penning trap. These kind of set ups provide a clean environment to test fundamental and delicate quantum predictions.

One of the basic postulate in quantum theory is that to every process there is a probability amplitude. It is known that if we prepare a system in the state $|\Psi_i\rangle$ and a measurement is performed to see if it is in a state $|\Psi_f\rangle$ at a later time, then the probability of transition from $|\Psi_i\rangle$ to $|\Psi_f\rangle$ is given by the modulus square of the transition probability amplitude. The transition amplitude between two non-orthogonal states is given by the complex number $\langle\Psi_i|\Psi_f\rangle$ whose modulus and phase (or argument) both have physical meaning. However, the issue of phase is always very subtle in the quantum world and in recent years our understanding of various phases has progressed. When we talk of phase of a state, it is the relative that is always important. The over all phase plays no role, because that gives a same ray in the Hilbert space. The relative phase difference between an initial and a final state in general contains a dynamical and a geometrical phase. In brief, dynamical phase corresponds to time integral of average of the Hamiltonian and geometric phase corresponds to a path dependent term caused by the non-trivial geometry of the state space. Under QZD the transition probability is affected giving rise to QZE. It is pertinent here to ask an intriguing question: what happens to the phase under QZD? This question was first realised by the present author. Specifically, we will be concerned with the evolution of the geometric phase under QZD. There are some deep reasons for this. First, in quantum theory the physical quantity of interest should not depend on a particular choice of gauge. The transition probability and geometric phase are in fact gauge invariant under $U(1)$ action of the state vectors. Second, both these quantities are ray space objects and hence they are on the same footing. It is thus natural to study the behaviour of the geometric phase and not all other phases under QZD.

If the final state has been obtained according to unitary and reversible dynamics, then the transition amplitude is $\langle\Psi_i|\Psi_f\rangle = \langle\Psi_i|U(t_f - t_i)|\Psi_i\rangle$. This evolution introduces non-cyclic dynamical phase and geometric phase. On the other hand if the state of the system follows non-unitary evolution due to sequence of measurements of *non-commuting observables* (hence the corresponding projectors also do not commute) then the system acquires a pure geometric phase. This type of geometric phase was first encountered by Pancharatnam [4]. The important point here is that if the successive measurements are carried out along shortest geodesics, then $|\Psi_i\rangle$ is in phase with $|a\rangle$ and $|a\rangle$ is in phase with $|b\rangle$ but $|b\rangle$ need not be in phase with $|\Psi_i\rangle$. This non-transitivity of phase preserving relation is often responsible for giving rise to geometric phase [5].

It is now well known that unitary evolution introduces dynamical as well as geometric phases and evolution due to projections (measurements by non-commuting projectors) introduce only geometric phase. What is the situation when both the evolutions are present? In this paper we discuss the issue of phases in quantum measurement theory and focus our attention mainly on the geometric phases. The paper is organised as follows. In Sec. 2 we briefly introduce the conecpts of total phase, dynamical phase and geometric phase. In Sec. 3 we investigate the behaviour of geometric phase when unitary evolution is interrupted by successive measurements of the von Neumann type. We predict that the geometric phase of the quantum system is also inhibited like the transition probability itself under quantum Zeno dynamics. The main result is that there is a finite, non-zero geometric phase with finite number of measurements which goes to zero under continuous measurement, thus erasing the memory of a quantum system. We may refer to this effect as *quantum Zeno Phase effect* (QZPE). In Sec. 4 we illustrate the effect with neutron spin undergoing rotation and projection. In Sect. 5 we we prove the QZPE based on a specific model of continuous quantum measurements. The dynamical equation is non-unitary and non-linear thereby invoking the irreversibility of the measurement process but in a sense the description was beyond von Neumann's (orthodox) quantum mechanics. In Sec. 6 we propose an interference experiment to test the quantum Zeon phase effect. The continuous measurement version is difficult to implement in laboratory whereas the discrete and instantneous model can be tested by means of suitable interference experiments with neutrons or photons. In Sec. 7 the conclusion follows. An important implication of this effect is that it provides a quantum mechanical way of controling and manipulating phase shifts. In recent years controlling phase of quantum systems play a very important role in quantum information processing, so this effect could be useful in a wide variety of context.

## 2 Total phase, Dynamical phase and Geometric Phase

Before discussing the quantum Zeno Phase effect (QZPE) we breifly mention what is all about the total phase, dynamical phase and geometric phase. Since we will be interested in quantum mechanical processes which involve both unitary and non-unitary operations, we will invoke the most general geometric phase defined by the present author. Therefore, we go beyond the one that was first discovered by Berry [6] in his pioneering work on quantum adiabatic theorem. First, we briefly sketch the adiabatic Berry phase. The Berry phase basically attributes a "memory" to the quantum system because it depends purely on the geometry of the evolution path thereby remembering its history. If we have a quantum system governed by an adiabatically changing Hamiltonian $H(\mathbf{R})$ (with non-degenerate eigenvalue $E_n(\mathbf{R})$ and eigenfunction $|\Psi_n(\mathbf{R(t)})\rangle$), then the wave fuction of the system acquires an extra phase given by

$$|\Psi(T)\rangle = e^{-\frac{i}{\hbar}\int_0^T E_n(t)\ dt} e^{i\gamma_n(C)}|\Psi_n(\mathbf{R(t)})\rangle, \tag{1}$$

where $\gamma_n(C) = i \oint \langle \Psi_n(\mathbf{R}(\mathbf{t})) | \nabla \Psi_n(\mathbf{R}(\mathbf{t}))\rangle . d\mathbf{R}$, is the Berry phase that depends only on the geometry of the path in the parameter space.

This concept was generalized to non-adiabatic, cyclic evolutions by Aharonov and Anandan [7]. Further, it was generalized to non-adiabatic, non-cyclic evolutions by Samuel and Bhandari [8]. A kinematic and group theoretic approach to geometric phase was provided by Mukunda and Simon [9]. More recently, it is shown by the present author [10, 11] that the geometric phase also arises for non-adiabatic, non-cyclic, non-unitary and non-Schrödinger evolutions of quantum systems. The adiabatic Berry phase and Hannay angle for open paths has been introduced by the present author in a recent paper [12].

There are also interesting developements on geometric phases by various authors in the context of measurement theory. Anandan and Pines [13] have considered a sequence of incomplete measurements and obtained a non-abelian geometric phase. Bhandari [14] has argued that in Einstein's gedanken experiment *for which path information* in a double-slit interference experiment the geometric phase plays the role of a random phase which washes out the interference pattern. Pati and Lawande [15] have proved that the development of the geometric phase can be made to zero under the limit of high rate of information gain about an observable. Berry and Klein [16] have studied the geometric phase when light beam passes through twisted stacks of N polarisers or retarders and found a finite non-zero geometric phase in the continuum limit. Aharonov *et al.* [17] have shown that the back reaction due to measurement process can induce a gauge potential in the system. Recently Pati and Lawande [18] have predicted the quantum Zeno Phase effect within the collapse mechanism, which we discuss in next section. It is quite amazing that when a sequence of measurements are performed on a quantum system one can get the inhibition of geometric phase in the limit of continuous measurement.

We briefly discuss the generalized geometric phase. Let us consider an arbitrary quantum system whose state vector $|\Psi(t)\rangle \in \mathcal{H}$ the Hilbert space of the system with finite or infinite dimension. The dynamical evolution is effected by a Hamiltonian $H$. The state vector of the system evolves according to Schrödinger equation (in the absence of measurement process)

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \tag{2}$$

Consider an arbitrary evolution (in general a non-cyclic) of a quantum system for a time $T$. The final state is $|\Psi(T)\rangle = U(T)|\Psi(0)\rangle$. We know that the natural way to define the phase difference between two non-orthogonal state is to take the inner product and then the argument of the inner product will give the total phase. This is the Pancharatnam way of defining the phase. The total phase is given by

$$\Phi_t = \arg\langle\Psi(0)|U(T)|\Psi(0)\rangle = \arg\langle\Psi(0)|\Psi(T)\rangle. \tag{3}$$

It was shown by the present author [10] that the total phase contains a dynamical phase and a geometric phase and the corresponding expressions are given, respectively, by

$$\begin{aligned} \Phi_d &= \frac{1}{\hbar}\int_0^T \langle\Psi(t)|H(t)|\Psi(t)\rangle \, dt, \\ \Phi_g &= i\int_0^T \langle\chi(t)|d\chi(t)\rangle, \end{aligned} \tag{4}$$

where $|\chi(t)\rangle$ is the reference-state of the system introduced in [10, 11], given by $|\chi(t)\rangle = \frac{\langle\Psi(t)|\Psi(0)\rangle}{|\langle\Psi(t)|\Psi(0)\rangle|}|\Psi(t)\rangle$. Here, $i\langle\chi(t)|d\chi(t)\rangle$ is a connection-form whose line integral along open path in the projective Hilbert space of the quantum system gives the desired geometric phase. This geometric phase is gauge invariant and does not depend on the detailed dynamics of the system. There could be an infinite number of open-paths in the Hilbert space but for all of them the geometric phase defined above is the same for a given projection of the open-path in the projective Hilbert space of the quantum system. This phase is, in general, *non-additive in nature* and assigns a memory to the quantum system. It is important to note that the Berry phase and Aharonov-Anandan phase are special case of the generliased geometric phase.

## 3 Quantum Zeno Phase Effect within von-Neumann's collapse postulate

In this section we discuss the quantum Zeno Phase effect using collapse postulate and unitary evolution. This has been recently predicted in [18] though the present author has conjectured it long back. Let us consider a quantum system where we have prepared it initially in one of the eigenstates of some observable $O$, which we are interested in measuring. The Hermitian operator $O$ has an eigenvalue spectrum $\{O_n\}$ and a complete set of eigenfunctions $\{|\Psi_n\rangle\}$. The spectrum is assumed to be discrete and non-degenerate. The observable $O$ need not commute with the Hamiltonian. We allow the system to evolve under the Hamiltonian $H$ during the time interval $[0, T]$ such that one performs a series of measurements at times $\tau, 2\tau, \ldots, (N-1)\tau, N\tau = T$. During the short time interval $[0, \tau]$ the system evolves unitarily. The sequence of measurements that are carried out are idealised to be discrete and instantaneous. We are interested in knowing how the geometric phase is affected due to $N$ number of measurements on the quantum system. The fact that the geometric phase is non-additive in nature implies that the sum of the geometric phases acquired by the system from time 0 to $\tau$, $\tau$ to $2\tau$ ... $(N-1)\tau$ to $N\tau$ is not the same as the geometric phase acquired by the system from 0 to $N\tau = T$, unlike the dynamical phase which is additive.

Here we proceed to give a simple derivation of the result where the dependence of the number of measurement on the geometric phase can be seen explicitly. Starting from an initial state $|\Psi_n\rangle$ the state at time $\tau$ is given by

$$|\Psi(\tau)\rangle = U(\tau)|\Psi_n\rangle = e^{-iH\tau/\hbar}|\Psi_n\rangle. \tag{5}$$

After performing a von Neumann measurement at time $\tau$ (denoted by a projection operator $P_n = |\Psi_n\rangle\langle\Psi_n|$) to know whether the eigenvalue of the observable is still $O_n$, the state is given by $|\Psi_{am1}(\tau)\rangle = P_n|\Psi_n\rangle = |\Psi_n\rangle\langle\Psi_n|U(\tau)|\Psi_n\rangle$, where $|\Psi_{am1}(\tau)\rangle$ is the state of the system just after first measurement at time $\tau$. Similarly, the state of the system at time $2\tau$ is $|\Psi(2\tau)\rangle = \langle\Psi_n|U(\tau)|\Psi_n\rangle\, U(\tau)|\Psi_n\rangle$. When we perform second instantaneous measurement, the state after the measurement is given by $|\Psi_{am2}(\tau)\rangle = \langle\Psi_n|U(\tau)|\Psi_n\rangle\langle\Psi_n|U(\tau)|\Psi_n\rangle)|\Psi_n\rangle$. If we proceed with $N$ number of measurement steps, the state of the system just after $N$th measurement is given by

$$|\Psi_{amN}(\tau)\rangle = \langle\Psi_n|U(\tau)|\Psi_n\rangle \ldots \langle\Psi_n|U(\tau)|\Psi_n\rangle|\Psi_n\rangle. \tag{6}$$

This is the final state of the system at time $T$ under quantum Zeno dynamics. Therefore, the total phase of the system can be obtained by taking the argument of the inner product of the initial and final state, in the sense of Pancharatnam. Thus, the total phase is given by

$$\Phi_t^{(N)}(T) = \arg\Big[\langle\Psi_n|U(\tau)|\Psi_n\rangle \ldots \langle\Psi_n|U(\tau)|\Psi_n\rangle\Big], \tag{7}$$

where inside the bracket we have $N$ such products of amplitudes $\langle\Psi_n|U(\tau)|\Psi_n\rangle$. If there is a complete set of eigenfunctions $\{|\Phi_m\rangle\}$ coresponding to the Hamiltonian $H$ with eigenvalue $\{E_m\}$, then we can write the each amplitude $\langle\Psi_n|U(\tau)|\Psi_n\rangle$ as

$$\langle\Psi_n|U(\tau)|\Psi_n\rangle = \sum_m e^{\frac{i}{\hbar}E_m t} P_{nm}, \tag{8}$$

where $P_{nm} = |\langle\Phi_m|\Psi_n\rangle|^2$ is called doubly-stochastic matrix, with the property that $\sum_n P_{nm} = \sum_m P_{nm} = 1$.

The dynamical phase (which is an additive quantity) of the system under quantum Zeno dynamics is given by

$$\begin{aligned}\Phi_d^{(N)}(T) &= -[\langle\Psi_n|H|\Psi_n\rangle\frac{\tau}{\hbar} + \ldots] = -\langle\Psi_n|H|\Psi_n\rangle N\frac{\tau}{\hbar} \\ &= -\frac{1}{\hbar}\langle\Psi_n|H|\Psi_n\rangle T = -\frac{1}{\hbar}\sum_m E_m P_{nm} T,\end{aligned} \tag{9}$$

where we have inserted a complete set of eigenfunctions of the Hamiltonian. The dynamical phase does not depend on the number of measurements performed on the quantum system under going unitary evolution. Therefore, the dynamical phase is insensitive to the quantum Zeno dynamics.

Therefore,the geometric phase $\Phi_g$ is can now be expressed as

$$\begin{aligned}\Phi_g^{(N)} &= \arg\Big[\langle\Psi_n|U(\tau)|\Psi_n\rangle \ldots \langle\Psi_n|U(\tau)|\Psi_n\rangle\Big] + \langle\Psi_n|H|\Psi_n\rangle\frac{T}{\hbar} \\ &= N\tan^{-1}\left[\frac{-\sum_m E_m P_{nm}\frac{T}{N\hbar}}{1-\sum_m E_m^2 P_{nm}\frac{T^2}{2N^2\hbar^2}}\right] + \sum_m E_m P_{nm}\frac{T}{\hbar}.\end{aligned} \tag{10}$$

The above expression clearly shows the *dependence of the geometric phase on the number of measurements* (i.e., the number of times the system's wave-function has been collapsed) that has been performed on the quantum system.

Now the interesting situation arises, when one takes the large $N$ limit, i.e., in the limit of continuous measurement, a continuous measurement is understood as the limit of a sequence of discrete, instantaneous measurements when the number of measurements tend to infinity. Then the geometric phase behaves as $\Phi_g^{(N)}$

$$\lim_{N\to\infty}\Phi_g(t) = \lim_{N\to\infty} N\tan^{-1}\left[\frac{-\sum_m E_m P_{nm}\frac{T}{N\hbar}}{1-\sum_m E_m^2 P_{nm}\frac{T^2}{2N^2\hbar^2}}\right] + \sum_m E_m P_{nm}\frac{T}{\hbar} \to 0, \tag{11}$$

where we have replaced $\tan^{-1} x = x$ for small $x$. This shows that continuous measurement in quantum Zeno dynamics setting can completely inhibit the development of the geometric phase. Since the geometric phase attributes a "memory" to the quantum system the above result shows that under quantum Zeno dynamics the "memory" of a system can be erased. It should be mentioned that QZPE is not a consequence of QZE. It is wrong to say that QZE implies stationarity of the quantum state and for a stationary state geometric phase is zero, so QZPE follows from QZE. We have shown here that under QZD neither the total nor the dynamical phase goes to zero. So QZPE is an independent, yet another non-trivial consequence of QZD. The prediction of *quantum Zeno Phase effect* QZPE requires only unitary Schrödinger evolution and the projection postulate. Recently, Pati and Lawande [19] have shown that the QZE in fact occurs for a wide class of systems, obeying non-linear, non-unitary equations and one may hope that the QZPE can also be predicted for non-linear systems.

## 4 Quantum Zeno Phase effect for neutron

In this section we illustrat the quantum Zeno Phase effect for the neutron spin undergoing "free evolution" and measurement of its spin in its initial state. Let us consider a source which sends a spin-up neutron that passes through several identical magnetic field regions. The magnetic field could have components along all the three directions $x, y$ and $z$. When the spin of neutron passes through the magnetic field, it undergoes precession. During precession the spin state can acquire a geometric phase in addition to the usual dynamical phase arising from instantaneous rotation. The interaction of the spin with the magnetic field $\mathbf{B}$ can be described by a Hamiltonian $H = \mu\sigma.\mathbf{B}$, where $\mu$ is the magnetic moment, $\sigma$ is the Pauli spin matrices. If the initial state is prepared in the state $|\Psi(0)\rangle = |\uparrow\rangle$, then the state after a time $t$ is given by

$$|\Psi(t)\rangle = e^{-\frac{i\mu\sigma.\mathbf{B}}{\hbar}t}|\uparrow\rangle = a(t)|\uparrow\rangle + b(t)|\downarrow\rangle. \tag{12}$$

Here, $a(t) = (\cos\frac{\omega t}{2} - in_z\sin\frac{\omega t}{2})$, $b(t) = (n_y - in_x)\sin\frac{\omega t}{2}$, $\omega = \frac{2\mu B}{\hbar}$ and $\mathbf{n} = (n_x, n_y, n_z)$ is the unit vector in the direction of the magnetic field. If the evolution is not interrupted by the measurement process the non-cyclic geometric phase is given by

$$\Phi_g(t) = -\tan^{-1}\left(n_z\tan\frac{\omega t}{2}\right) + n_z\frac{\omega t}{2}. \tag{13}$$

which can be interpreted as half the solid angle subtended by the closed curve obtained by joining the end points of the open curve with a shortest geodesic. The open curve can be parametrised by the azimuthal angle $\phi = \omega t$ and polar angle $\theta = \cos^{-1}(n_z)$, on a sphere which is the projective Hilbert space for the neutron.

Let us now investigate the situation when the neutron spin is monitored $N$ number of times during its evolution over a time period $T$ and under going precession in several magnetic field regions each with a length $l$ (here $l$ is very small). Let there be a device to select and detect the spin component of the neutron as in the proposed experiment to test the QZE by Nakazato *et al.* [20–22]. When neutron passes through a magnetic field region for a small time $\tau = l/v$, with $v$ being the speed of neutron in the field region), it undergoes rotation. The state of the neutron after passing through sequence of rotation and projection is $|\Psi_{amN}(N\tau)\rangle = a(\tau)^N|\uparrow\rangle$. Now the geometric phase shift relative to the initial state during a quantum Zeno dynamics process is given by

$$\Phi_g(N) = -N\tan^{-1}\left(n_z\tan\frac{\omega T}{2N}\right) + n_z\frac{\omega T}{2}. \tag{14}$$

In the limit of large number of measurements the geometric phase for neutron spin goes to zero, demonstrating the quantum Zeno Phase effect, in conformity with our prediction.

## 5 Quantum Zeno Phase effect within a continuous measurement model

In this section we prove the quantum Zeno Phase effect using a continuous measurement model. Unlike the von-Neumann's collapse mechanism, the present model does not assume the existence of projection operator which can bring an instantaneous change in the wavefunction. There are various models of continuous measurement, where one treates system and the apparatus both quantum mechanically. Further, if the apparatus is coupled to environment, then one can describe the mechanism of collapse through what is now called "decoherence" process. One could, for example, use the Lindbladian formalism [23] to describe the evolution of the quantum system through a reduced density matrix [24] and see how the geometric phase of the quantum system would change as the decoherence rate increases. However, this approach would need the knowledge of quantum phase for mixed states undergoing non-unitary, non-cyclic evolution, which we belive not yet defined. So, we take a different approach to continuous meaurement model. We consider the system to be quantum mechanical, the apparatus to be classical, and the interaction of system and apparatus is included in a phenomenological way. The system can be either be described by a wavefunction or a (pure) density matrix (because we have not integrated over the rest of the universe). The cost we have to pay is that the effective Hamiltonian of the sytem is now an non-Hermitian operator, which it has to be in order that the irreversibility arises during measurement process. We propose that the density operator $\rho(t)$ of the quantum system obeys an equation

$$i\hbar\frac{d\rho(t)}{dt} = [H_0, \rho(t)] - i\{H_1, \rho(t)\} + 2i\mathrm{Tr}(H_1\rho(t))\rho(t), \tag{15}$$

where $[H_0, \rho(t)]$ is the commutator and $\{H_1, \rho(t)\}$ is the anti-commutator between corresponding operators. Here, $H_0$ is the free Hamiltonian of the system and $H_1$ is the interaction Hamiltonian which depends on the observables of the system, measurement apparatus coordinate etc. We can take in general $H_1$ to be

$$H_1 = H_1(A; \Delta a, a, f) = -\hbar f \;\; G(A; \Delta a, a) \tag{16}$$

where $A$ is the observable of the system that is being measured, $\Delta a$ is the accuracy with which the device records the observable, $a$ is the result of the measurement and $f$ is the freqeuncy of the meausurement process (the number of measurements per unit time interval). The exact form of the parametric operator $G(A; \Delta a, a)$ is not specified but any positive and bounded oprator would bring an irreversible change in the quantum system. The above equation is non-linear one but it preserves trace and positivity of the density matrix. In fact, the term which makes the equation non-linear, is responsible for preserving the traces during the time evolution.

If we have a pure state density operator $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ and take a specific form of the Hamiltonian $H_1 = -\hbar f \;\; G(\frac{A-a}{\Delta a})$, then we can write an equation of motion for the state vector $|\Psi(t)\rangle$ given by

$$i\hbar\frac{d|\Psi(t)\rangle}{dt} = H_0|\Psi(t)\rangle - i(H_1 - \langle\Psi(t)|H_1|\Psi(t)\rangle)|\Psi(t)\rangle. \tag{17}$$

This equation is equivalent to the one investigated by Kulaga [24] and derived using restricted path integral approach [25, 26] in the context of quantum Zeno effect. We use our density matrix equation to prove the quantum Zeno Phase effect. Let us consider the measurement of observable $A$ which has a non-degenrate spectrum $\{a_n\}$ and a complete set of eigenstates $\{|\psi_n\rangle\}$. We assume for simplicity that the operator $A$ commutes with the free Hamiltonian. With the above condition, we find the solution to (15) as

$$\rho(t) = \frac{1}{\sum_{nm} e^{-2fG_n t}\rho_{nn}(0)} \sum_{nm} e^{-\frac{i}{\hbar}(E_n - E_m)t} e^{-f(G_n+G_m)t} \rho_{nm}(0)|\psi_n\rangle\langle\psi_m|, \quad (18)$$

where $E_n$ 's are eigenvalues of the free Hamiltonian, $G_n = G(a_n; \Delta A, a)$. To calculate the geometric phase, we need to express the generalized geometric phase in terms of the density operator $\rho(t)$. Interestingly, such an expression has been be found by Wagh and Rackhecha [28] and is given by

$$\Phi_g(t) = \frac{i}{2}\int dt\, \frac{\mathrm{Tr}(\rho(0)[\rho(t), \dot{\rho}(t)])}{\mathrm{Tr}(\rho(0)\rho(t))}, \quad (19)$$

which can also be obtained from our connection-form formula (4). The above formula throughs light on the quantum mechanical origin of the geometric phase. It is arises due to the non-commuting nature of the density operator and its differential. Since commutator of two Hermitian operator is a anti-Hermitian operator, we can write the above one as

$$\Phi_g(t) = \frac{1}{2}\int dt\, \frac{\mathrm{Tr}(\rho(0)C(t))}{\mathrm{Tr}(\rho(0)\rho(t))} = \frac{1}{2}\int dt\, \frac{\sum_{nm}\rho_{nm}(0)C_{nm}(t)}{\sum_{nm}\rho_{nm}(0)\rho_{nm}(t)}, \quad (20)$$

where $C(t) = i[\rho(t), \dot{\rho}(t)]$ is a Hermitian operator. On writting the density operator as $\rho(t) = \sum_{nm}\rho_{nm}(t)|\psi_n\rangle\langle\psi_m|$, we can find the matrix elements of $C(t)$. It is given by

$$\begin{aligned} C_{nm}(t) &= i\sum_k(\rho_{nk}(t)\dot{\rho}_{km}(t) - \dot{\rho}_{nk}(t)\rho_{km}(t)) \\ &= \sum_k[2E_k - (E_n + E_m)]\rho_{nk}(t)\rho_{km}(t), \end{aligned} \quad (21)$$

where we have used the Hermiticity of the matrix elements in obtaining the above expression. Inserting the above equation into (20) and using (18) we obtain the geometric phase formula during a continuous measurement process, given by

$$\begin{aligned} \Phi_g(t) &= \frac{1}{2}\int dt\, \frac{\sum_{nmk}[2E_k - (E_n+E_m)]\rho_{nk}(t)\rho_{km}(t)\rho_{nm}(0)}{\sum_{nm}\rho_{nm}(0)\rho_{nm}(t)} \quad (22) \\ &= \frac{1}{2}\int dt\, \frac{1}{\sum_{nm} e^{-2fG_n t}\rho_{nn}(0)} \\ &\times \frac{\sum_{nmk}[2E_k - (E_n+E_m)]e^{-\frac{i}{\hbar}(E_n-E_m)t}e^{-f(G_n+G_m+2G_k)t}\rho_{nn}(0)\rho_{mm}(0)\rho_{kk}(0)}{\sum_{nm} e^{-\frac{i}{\hbar}(E_n-E_m)t}e^{-f(G_n+G_m)t}\rho_{nn}(0)\rho_{mm}(0)} \end{aligned}$$

In the limit of of high rate of information gain on the observable $A$, we have $f \to \infty$. In this limit, the summations will reduce to single term for some $n, m, k = \bar{n}$, if $E_{\bar{n}}$ = min $\{E_n\}$

and $G_{\bar{n}}$ = min $\{G_n\}$ and similarly for other indices. It can be seen from (22) that the when the limit is taken, the integrand (namely, the connection-form) goes to zero, i.e.

$$\lim_{f\to\infty} \Phi_g(t) \to \frac{1}{2}\int dt \frac{2[E_{\bar{n}} - E_{\bar{n}})\rho_{\bar{n}\bar{n}}(0)}{\rho_{\bar{n}\bar{n}}(0)} = 0. \tag{23}$$

This proves that when we perform infinity number of measurements on an observable then the geometric phase of the quantum system can go to zero, which is nothing but the inhibition of the quantum phase. But this time, we have proved it within a continuous measurement model without collapse postulate. We conjecture that the quantum Zeno Phase effect can also be proved within any model of quantum measurement. The effect seems to be independent of any particular measurement model.

## 6 A Proposal to test Quantum Zeno Phase Effect in experiment

A paradigm set up to measure the phase of a quantum state is through interference experiments. Here we propose a simple intereference experiment to test the quantum Zeno Phase effect using neutron. The idea can equally be applied for light passing through a sequence of polarisers and analysers. Recently the geometric phase has been measured experimentally by Wagh *et al.* [29,30] for neutron spin under going precession. Here, we sketch briefly a neutron interference experiment to observe this new effect (QZPE). Let there be an incident neutron beam which is polarised along up direction and the beam is split coherently into two parts. In one arm (say 1) of the interferometer let there be several regions of magnetic field and $N$ number of detection devices. In the other arm (say 2) of the interferometer let there be a magnetic field present over a distance $L$ such that $L = vT$. The magnetic field is applied only along $z$-direction (with a magnitude equal to $B\cos\theta$) just to compensate the dynamical phase shift in the interference pattern. The state of the neutron passing through the arm 1 of the interferometer is $|\psi_1\rangle = K|\uparrow\rangle = a(\tau)^N|\uparrow\rangle$, where $K$ denotes the sequence of rotation and projection operation. The state in the arm 2 is $|\psi_2\rangle = \exp(-i\mu B\cos\theta\sigma_z T/\hbar)|\uparrow\rangle$. When the two neutron beams are recombined the intensity of the beam is given by

$$\begin{aligned} I(N) &\propto |||\psi_1\rangle + |\psi_2\rangle||^2 \\ &\propto \Big[(1 + |a(\tau)|^{2N}) + 2|a(\tau)|^N \cos(\Phi_g(N))\Big]. \end{aligned} \tag{24}$$

This shows the interference pattern is not simply of the type $1 + \cos\phi$. The modification in the first term is due to the non-unitary operation in the arm 1. The interference contrast and the phase shift of geometric origin depends on the number of measurements that have been performed in the arm 1 of the interferometer. This is an interesting observation and by changing the number of measurements we can have a different interference pattern. In the limit of large number of measurements $|a(\tau)|^{2N} = \exp[-(n_x^2 + n_y^2)\frac{\omega^2T^2}{4N}] \to 1$ and the geometric phase shift tends to zero, thus giving us a full maximum in the interference pattern- a consequence of the quantum Zeno Phase effect. (Interestingly, in the same set up one can test quantum Zeno effect also.)

## 7 Conclusion

In conclusion we have discussed various phase within quantum measurement theory. In particular, we reviewed one of our recent prediction on a new effect (Quantum Zeno Phase Effect), which says that under frequent measurements the geometric phase of any quantum system can be inhibited. This effect can be proved using collapse mechanism as well as any continuous measurement model. In future one may try to prove the QZPE using a particular model of decoherence. Interestingly, we have shown in this paper that *under a QZD neither the dynamical nor the total phase goes to zero* (though the later is affected under the number of measurements). Only the *geometric phase goes to zero* in the limit of continuous measurements when the system follows a QZD. The non-trivial result is that both the dynamical and total phase remains finite and geometric phase goes to zero in the limit of continuous measurements. More importantly, we have shown that under a QZD *there is a finite geometric phase* which goes to zero in the limit of large number of measurement pulses. This in turn implies that repeated measurements can erase the "memory" of a quantum system. We have illustrated the idea for a neutron spin under going successive rotations and projections. Also we have suggested a method to observe this effect in neutron interference experiments. This can provide a new way of controlling phase shift in quantum interference set ups. We hope with the present available technology it should be possible to test the quantum Zeno Phase effect.

**Acknowledgments** I wish to thank S. V. Lawande for encouragments and supports with whom some part of the work had been carried out in the past.

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