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EQUILIBRIUM OF SPINNING TEST PARTICLES IN THE SCHWARZSCHILD-DE SITTER SPACETIMES

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The possibility of equilibrium of test particles with spin located in the Schwarzschild-de Sitter spacetimes is investigated. The stationary equilibrium conditions resulting from the equations of motion and the equations of spin dynamics have to be satisfied simultaneously. It is shown that the equilibrium conditions are independent of the spin of the test particles; they are satisfied only at the so called static radius where the gravitational attraction is compensated by the cosmological repulsion.

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1 Introduction

Motion of test particles describes in an illustrative way properties of black-hole and naked-singularity backgrounds. The motion of uncharged and spinless test particles is governed by the geodesic equations and directly determines the geodesic structure of the spacetimes. Charged test particles can test combined gravitational and electromagnetic field of these backgrounds, the motion is determined by the Lorentz equation.

If the test particles possess also spin, their equations of motion must be much more complex in comparison with the spinless particles, because of the interaction of the spin with the curvature of the spacetime given by the Riemann tensor. Moreover, the spin dynamics has also to be considered. In the absence of an electromagnetic field of the background, the spin dynamics is determined by the Fermi-Walker transport equation [1].

Studies of the equilibrium positions of charged test particles give direct information on the interplay of the gravitational and electromagnetic forces acting in the charged (Reissner-Nordström and Kerr-Newman) backgrounds. The equilibrium of test particles in the Reissner-Nordström backgrounds was discussed by Bičák, Stuchlík and Balek [2], and independently by Bonnor [3]. They established basic general-relativistic modifications of the simple neutral (or separation independent) Newtonian equilibrium condition. The equilibrium of charged particles on the axis

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of symmetry of the Kerr-Newman backgrounds has been investigated in [2]. Existence of equilibrium of particles in the equatorial plane of the Kerr-Newman background has been indicated in [4], and exactly established in [5].

In the simplest, Schwarzschild backgrounds the equilibrium of test particles is impossible, because only gravitational attraction is acting there. However, recent cosmological observations give a strong evidence for the presence of a non-zero repulsive cosmological constant, $\Lambda > 0$, in the recent universe [6]. In the Schwarzschild-de Sitter spacetimes, even the equilibrium of uncharged particles is possible, due to the effect of the cosmological repulsion. Here, we shall study equilibrium of spinning test particles in the Schwarzschild-de Sitter backgrounds. Such an equilibrium is, of course, possible only in the spacetimes containing a static part [7].

The paper is organized in the following way. In Section 2, the Riemann curvature tensor is given for the Schwarzschild-de Sitter spacetimes. In Section 3, the equations of motion, and the spin dynamics equations are written down. In Section 4, the equilibrium conditions are established, and in Section 5 some concluding remarks are presented.

2 The Riemann Curvature Tensor

In the standard Schwarzschild coordinates (t, r, θ, ϕ) and the geometric units ($c = G = 1$), the Schwarzschild-de Sitter geometry with mass parameter M is determined by the line element

$$ds^2 = -\mathrm{e}^{2\Phi}\, dt^2 + \mathrm{e}^{2\Psi}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where

$$\mathrm{e}^{2\Phi(r)} = \mathrm{e}^{-2\Psi(r)} = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right). \tag{2}$$

Using the orthonormal tetrad of differential forms

$$\boldsymbol{e}^{(t)} = \mathrm{e}^{\Phi}\, dt, \quad \boldsymbol{e}^{(r)} = \mathrm{e}^{\Psi}\, dr, \quad \boldsymbol{e}^{(\theta)} = r\, d\theta, \quad \boldsymbol{e}^{(\phi)} = r \sin\theta\, d\phi, \tag{3}$$

and the dual tetrad of vectors

$$\boldsymbol{e}_{(t)} = \mathrm{e}^{-\Phi} \frac{\partial}{\partial t}, \quad \boldsymbol{e}_{(r)} = \mathrm{e}^{-\Psi} \frac{\partial}{\partial r}, \quad \boldsymbol{e}_{(\theta)} = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \boldsymbol{e}_{(\phi)} = \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}, \tag{4}$$

the Riemann curvature tensor can be found by using the standard method of the differential forms and the Cartan structure equations [1]. The first Cartan equation can be given in the form

$$0 = d\boldsymbol{e}^{(a)} + \omega^{(a)}{}_{(b)} \wedge \boldsymbol{e}^{(b)}, \tag{5}$$

with $\omega^{(a)}{}_{(b)}$ being the connection forms. For the orthonormal basis it follows from

$$dg_{(a)(b)} = \omega_{(a)(b)} + \omega_{(b)(a)} \tag{6}$$

that the connection form must be antisymmetric. The second Cartan structure equations relate the connection forms and the curvature forms $\Omega^{(a)}{}_{(b)}$:

$$\Omega^{(a)}{}_{(b)} = d\omega^{(a)}{}_{(b)} + \omega^{(a)}{}_{(c)} \wedge \omega^{(c)}{}_{(b)}. \tag{7}$$

Here d means exterior derivative, and $\wedge$ exterior product.

The tetrad components of the Riemann tensor $R^{(a)(b)}{}_{(c)(d)}$ are then determined by the formula

$$\Omega^{(a)(b)} = R^{(a)(b)}{}_{|(c)(d)|}\,\omega^{(c)} \wedge \omega^{(d)}; \tag{8}$$

$|(c)(d)|$ means that the sum over (c) and (d) is restricted to $(c) < (d)$, so each index pair occurs only once.

The coordinate components of the Riemann tensor $R^{\alpha\beta}{}_{\gamma\delta}$ are given by

$$R^{\alpha\beta}{}_{\gamma\delta} = e^{\alpha}_{(a)} e^{\beta}_{(b)} e^{(c)}_{\gamma} e^{(d)}_{\delta} R^{(a)(b)}{}_{(c)(d)}. \tag{9}$$

The tetrad components of the Riemann tensor can easily be found by the method presented in [1]. Due to the diagonality of the metric tensor of the Schwarzschild-de Sitter spacetimes in the Schwarzschild coordinate frame, and the diagonality of the electric and magnetic parts of the Riemann tensor in this frame, the tetrad and coordinate components of the Riemann tensor coincide if they are mixed (one pair up, the other down). The nonzero components are the following:

$$\begin{aligned} R^{tr}{}_{tr} &= R^{(t)(r)}{}_{(t)(r)} = R^{\theta\phi}{}_{\theta\phi} = R^{(\theta)(\phi)}{}_{(\theta)(\phi)} = \frac{2M}{r^3} + \frac{\Lambda}{3}, \\ R^{t\theta}{}_{t\theta} &= R^{(t)(\theta)}{}_{(t)(\theta)} = R^{t\phi}{}_{t\phi} = R^{(t)(\phi)}{}_{(t)(\phi)} = R^{r\theta}{}_{r\theta} = R^{(r)(\theta)}{}_{(r)(\theta)} \\ &= R^{r\phi}{}_{r\phi} = R^{(r)(\phi)}{}_{(r)(\phi)} = \frac{\Lambda}{3} - \frac{M}{r^3}. \end{aligned} \tag{10}$$

3 The Equations of Motion and Spin Dynamics

The motion of a spinning test particle in an arbitrary gravitational field has been studied by Papapetrou [8]. Such a particle deviates from its geodesic and moves along a different orbit according to

$$\frac{D}{d\tau}\left(mu^{\alpha} + u_{\beta}\frac{DS^{\alpha\beta}}{d\tau}\right) + \frac{1}{2}R^{\alpha}{}_{\nu\sigma\mu}u^{\sigma}S^{\mu\nu} = 0, \tag{11}$$

while its spin evolves according to

$$\frac{DS^{\alpha\beta}}{d\tau} + u^{\alpha}u_{\rho}\frac{DS^{\beta\rho}}{d\tau} - u^{\beta}u_{\rho}\frac{DS^{\alpha\rho}}{d\tau} = 0, \tag{12}$$

where m and u^{α} are the particle's mass and 4-velocity, $D/d\tau$ denotes a covariant derivative along u^{α}, $S^{\alpha\beta}$ is the antisymmetric spin tensor of the particle, $R^{\alpha}{}_{\nu\sigma\mu}$ is the Riemann tensor describing the background on which the particle moves. The test particle is assumed so small in size and in mass that it does not modify the background, and the radiation reaction has a negligible effect on the particle's orbit. Clearly, for vanishing spin the particle moves along a geodesic.

By introducing the Pirani spin supplementary condition [9]

$$S^{\alpha\beta}u_{\beta} = 0 \tag{13}$$

and the covariant spin vector

$$S_\sigma \equiv \frac{1}{2}\epsilon_{\rho\mu\nu\sigma}u^\rho S^{\mu\nu}, \tag{14}$$

the Papapetrou equations of motion can be transferred into the following form [1]

$$m\frac{Du^\alpha}{d\tau} = -\epsilon^{\alpha\mu\nu\beta}\frac{D^2u_\beta}{d\tau^2}S_\mu u_\nu + \frac{1}{2}\epsilon^{\lambda\mu\rho\sigma}R^{\alpha\nu}{}_{\lambda\mu}u_\nu u_\sigma S_\rho; \tag{15}$$

$\epsilon^{\alpha\beta\gamma\delta}$ is the completely antisymmetric Levi-Civita tensor. The replacement of the antisymmetric tensor $S^{\alpha\beta}$ by the vector S_α enables us to have a better understanding of the spin precession, since the space components S^i are the components of what we are used to think of as the spin 3-vector. Notice that by construction of the spin vector (see Eq. (14)), S_α is permanently orthogonal to the 4-velocity u^α:

$$S_\alpha u^\alpha = 0. \tag{16}$$

Dynamics of the spin vector is then given by a relatively simple equation of the Fermi-Walker transport (see [1])

$$\frac{DS_\alpha}{d\tau} = u_\alpha\frac{Du^\beta}{d\tau}S_\beta. \tag{17}$$

4 Equilibrium Conditions

In order to consider the possibility of equilibrium of a spinning test particle in the Schwarzschild-de Sitter spacetimes, we must find conditions which guarantee that the equations of motion (15), and the equations of spin dynamics (17), along with the orthonormality relation (16), are simultaneously satisfied for the 4-velocity u^α corresponding to a stationary particle in the background under consideration.

Since the test particle is at rest, its 4-velocity is given by the relations

$$u^\alpha = \frac{1}{\sqrt{-g_{tt}}}\,\delta^\alpha_t = \mathrm{e}^{-\Phi}\,\delta^\alpha_t, \quad u_\alpha = -\mathrm{e}^{\Phi}\,\delta^t_\alpha. \tag{18}$$

Then we also find that for the stationary equilibrium of the test particles there is

$$\frac{du^\alpha}{d\tau} = u^\alpha{}_{,\beta}u^\beta = 0. \tag{19}$$

¿From the orthogonality relation (16) it follows that

$$S_t = 0, \tag{20}$$

and the spin of the stationary particles has space components only. Because of the form of u^α and S_α, the equation of the spin dynamics (17) reduces to

$$\frac{dS_\alpha}{d\tau} = \left[\Gamma^i{}_{\alpha t}u^t + u_\alpha\Gamma^i{}_{tt}(u^t)^2\right]S_i, \tag{21}$$

where $\Gamma^{\mu}{}_{\nu\kappa}$ denote coefficients of the affine connection of the Schwarzschild-de Sitter spacetimes. Using the explicit form of the coefficients, we arrive to the relation

$$\frac{dS_{\alpha}}{d\tau} = 0. \tag{22}$$

Therefore, the spinning test particles located in the static Schwarzschild-de Sitter spacetimes in an equilibrium stationary position must have their spin vector fixed. It is a stronger condition than the constancy of the square of the spin

$$\frac{d}{d\tau}(S_{\alpha}S^{\alpha}) = 0, \tag{23}$$

which is then satisfied automatically.

Now, we consider the equations of motion (15). First, we obtain relation governing the second derivative of the 4-velocity. Generally, there is

$$\begin{aligned}\frac{D^2 u_{\alpha}}{d\tau^2} &= u_{\alpha;\beta\gamma}u^{\beta}u^{\gamma} + u_{\alpha;\beta}u^{\beta}{}_{;\gamma}u^{\gamma}\\ &= u_{\alpha,\beta\gamma}u^{\beta}u^{\gamma} + u_{\alpha,\beta}u^{\beta}{}_{,\gamma}u^{\gamma} - \Gamma^{\mu}{}_{\alpha\beta,\gamma}u_{\mu}u^{\beta}u^{\gamma}\\ &\quad - 2\Gamma^{\mu}{}_{\alpha\beta}u_{\mu,\nu}u^{\gamma}u^{\beta} - \Gamma^{\mu}{}_{\alpha\beta}u^{\beta}{}_{,\gamma}u^{\gamma}u_{\mu}\\ &\quad + \Gamma^{\mu}{}_{\alpha\gamma}\Gamma^{\kappa}{}_{\mu\beta}u_{\kappa}u^{\beta}u^{\gamma}.\end{aligned} \tag{24}$$

For the stationary particles with u^{α} and $du^{\alpha}/d\tau$ given by the relations (18) and (19), we find

$$\frac{D^2 u_{\alpha}}{d\tau^2} = -u^t\left(\Gamma^{t}{}_{\alpha t}\Gamma^{t}{}_{tt} + \Gamma^{r}{}_{\alpha t}\Gamma^{t}{}_{rt} + \Gamma^{\theta}{}_{\alpha t}\Gamma^{t}{}_{\theta t} + \Gamma^{\phi}{}_{\alpha t}\Gamma^{t}{}_{\phi t}\right), \tag{25}$$

which for the Schwarzschild-de Sitter backgrounds reduces to

$$\frac{D^2 u_{\alpha}}{d\tau^2} = -u^t\,\Gamma^{r}{}_{\alpha t}\Gamma^{t}{}_{rt}; \tag{26}$$

it has only one non-zero component:

$$\frac{D^2 u_t}{d\tau^2} = -u^t\,\Gamma^{r}{}_{tt}\Gamma^{t}{}_{rt} = -\frac{(M-\Lambda r^3/3)^2}{r^{7/2}(r-2M-\Lambda r^3/3)^{1/2}}. \tag{27}$$

However, we can immediately see that this component cannot enter the equations of motion (15) because of the completely antisymmetric Levi-Civita tensor. Further, we can easily convince ourselves that also the Riemann curvature tensor will not enter the equations of motion of spinning particles in the Schwarzschild-de Sitter background. Therefore, we arrive to rather surprising result that in the static Schwarzschild-de Sitter spacetimes the equations of motion (15) are not influenced by the spin vector of the stationary spinning particles. They are reduced to geodesic equations

$$\frac{Du^{\alpha}}{d\tau} = 0. \tag{28}$$

For the stationary particles we then obtain the condition

$$\Gamma^{r}{}_{tt}\left(u^t\right)^2 = 0, \tag{29}$$

which implies

$$-\frac{M}{r^2} + \frac{\Lambda}{3} r = 0. \tag{30}$$

The equilibrium is, therefore, possible just at the so called static radius

$$r = r_{\mathrm{s}} \equiv \left(\frac{3M}{\Lambda}\right)^{1/3}. \tag{31}$$

5 Concluding Remarks

We have found that, surprisingly, stationary equilibrium of a spinning test particle in the Schwarzschild-de Sitter backgrounds is independent of the mass and spin of the test particle, and it is allowed only at the static radius where the gravitational attraction of the black hole acting on the particle is just compensated by the cosmological repulsion. The spin vector of the particle can be arbitrary, and it will be time-independent.

In the Schwarzschild-anti-de Sitter spacetimes with an attractive cosmological constant, $\Lambda < 0$, the equilibrium of both spinning and spinless particles is forbidden because both the mass of the black-hole and the cosmological constant have attractive character.

If rotation of the background is "switched on", we obtain a spin-dependent behavior of the stationary equilibrium of a spinning test particle, because of the coupling between the angular momentum of the background and the spin of the particle. Such a situation can occur, e.g., in the Kerr-de Sitter spacetimes, where the equilibrium is possible due to the effects of cosmological repulsion [10].

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