acta physica slovaca vol. 49 No. 2, 185 – 192 April 1999

ELEMENTS OF A FIELD THEORY ON A NONCOMMUTATIVE CYLINDER*

H. Grosse
Institut for Theoretical Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

P. Prešnajder[1]
Department of Theoretical Physics, Comenius University Mlynská dolina, SK-84215 Bratislava, Slovakia

Received 30 September 1998, in final form 2 January 1999, accepted 8 January 1999

First we briefly describe the scalar fields on a two dimensional space-time modeled as a standard (commutative) cylinder. Then we construct a noncommutative cylinder, a noncommutative generalization of a noncompact manifold. Free scalar field on a noncommutative cylinder is investigated. Due to the noncommutativity of the underlying space-time the energy frequencies are restricted, the time evolution is discrete. Within the perturbation theory this leads to an UV-regularization of Feynman diagrams. We expect that analogous properties will survive in a more realistic field theoretical models formulated on a noncommutative space-time.

PACS: 03.70

1 Introduction

The noncommutative geometry was developed by A. Connes in [1], [2], and applied to the standard model of fundamental interaction, see e.g. [3]. The essence of the noncommutative geometry consists first in reformulating the geometry in terms of commutative algebras and modules of smooth functions, and then generalizing them to their noncommutative analogs. In the applications mentioned above the ideas of the noncommutative geometry were used for the description of internal degrees of the model only. Even this led to a nontrivial restriction of admissible internal symmetries. Since the space-time was interpreted in a standard way as a continuum, this approach did not lead to the expected UV-regularization of the model. This would require a noncommutative model of the space-time.

If the notions of the noncommutative geometry are used directly for the desription of the space-time, the notion of points as elementary geometrical entity is lost and an ultraviolet cut-off is introduced. The simplest model of this kind is the Fuzzy sphere. It was introduced in

*Dedicated to Ján Pišút on the occasion of his 60th birthday.
[1] E-mail address: presnajder@fmph.uniba.sk

various contexts, [6-9]. In [10] - [13] various field theoretical models were describeded on a noncommutaive sphere. The noncommutativity of the base manifold led to the expected UV-regularization. However, the base space was compact and the models were formulated in the framework of Euclidean field theory.

Our aim is to be more physical. We shall desribe the scalar field on a manifold which has a clear notion of space and time. One of the simplest examples of such manifolds is the 2D cylinder $\mathbf{R} \times S^1$: the real line $\mathbf{R}$ is interpreted as the time and the space is identified with the circle S^1. The important new aspect here consists in the fact that the commutative manifold in question is noncompact, and this requires various modifications of the approach developed in the compact case, [14].

In Sec. 2 we collect relevant formulas on a scalar field on a standard commutative cylinder. The noncummutative version of the model is presented in Sec. 3. Last Sec. 4 contains concluding remarks and we sketch possible generalizations.

2 The commutative case

We identify in the standard (commutative) case the 2D cylinder C with the subset of points

$$C \ = \ \mathbf{R} \times S^1 \ = \ \{(t, x_1, x_2) \in \mathbf{R}^3 \ ; \ x_1^2 + x_2^2 = \rho^2\} \ . \tag{1}$$

We shall use the notation $x_\pm = x_1 \pm i x_2 = \rho e^{\pm i\phi}$.

Any complex function (scalar field) $\Phi(t, x)$, $x = (x_+, x_-)$, on C can be expanded as

$$\Phi(t,x) \ = \ a_0(t) \ + \ \sum_{k=1}^{\infty} a_k(t) x_+^k \ + \ \sum_{k=1}^{\infty} a_{-k}(t) x_-^k \ . \tag{2}$$

We introduce the standard integral on C by

$$I_0[\Phi] \ = \ \int_C dt dx \Phi(t,x) \ = \ \int_{\mathbf{R}} dt \, a_0(t) \ . \tag{3}$$

The corresponding field action we define in a usual way as

$$S[\Phi, \Phi^*] \ = \ I_0[\Phi^*(\Box - m^2)\Phi \ - \ V(\Phi^*\Phi)] \ . \tag{4}$$

Here $\Box = -\partial_t^2 + \partial_\varphi^2$ is the wave operator on C and $V(\Phi^*\Phi) = g(\Phi^*\Phi)^2 + \ldots$ is a suitable higher order polynomial bounded from below. Using expansion (2) we obtain

$$S[\Phi, \Phi^*] \ = \ S_0[\Phi, \Phi^*] \ + \ S_I[\Phi, \Phi^*] \ , \tag{5}$$

with $S_I[\Phi, \Phi^*] = I_0[V(\Phi^*\Phi)]$ and

$$S_0[\Phi, \Phi^*] \ = \ \int_{\mathbf{R}} dt \ \sum_{k \in \mathbf{Z}} [\dot{a}_k^*(t) \dot{a}_k(t) \ - \ (k^2 + m^2) a_k^*(t) a_k(t)] \ . \tag{6}$$

We now rewrite the model described by eq. (4) in algebraic terms, reinterpreting the cylinder C as the coadjoint orbit of the group $E(2)$ (of isometries of the two dimensional Euclidean plane). The corresponding $e(2)$ Lie algebra generators, X_0 (rotations) and $X_\pm$ (translations), satisfy commutation relations

$$[X_0, X_\pm] \;=\; \pm X_\pm \;,\; [X_+, X_-] \;=\; 0 \;. \tag{7}$$

On C the Lie-Kirillov Poisson bracket can be introduced: for any pair of functions Φ_1 and Φ_2 on C we put

$$\{\Phi_1, \Phi_2\} \;=\; \frac{\partial \Phi_1}{\partial x_\mu}\frac{\partial \Phi_2}{\partial x_\nu}\{x_\mu, x_\nu\} \;, \tag{8}$$

where the elementary brackets $\{x_\mu, x_\nu\}$, defined by

$$\{x_0, x_\pm\} \;=\; \pm i x_\pm \;,\; \{x_+, x_-\} \;=\; 0 \;. \tag{9}$$

We stress that the function x_+x_- is central: $\{x_\mu, x_+x_-\} = 0$. The restriction $x_+x_- = \rho^2$ is consistent with the Poisson bracket structure.

One can straightforwardly show that

$$\{x_\pm, \{x_\mp, \Phi\}\} \;=\; \partial_t^2 \Phi \;,\; \{x_0, \{x_o, \Phi\}\} \;=\; \partial_\varphi^2 \Phi \;. \tag{10}$$

Therefore, the wave operator $\Box$ can be expressed in terms of Poisson brackets as

$$\Box\Phi \;=\; \{x_0, \{x_0, \Phi\}\} \;-\; \{x_+, \{x_-, \Phi\}\} \;. \tag{11}$$

We finish this section with a few remarks concerning free field case. In this case $V(\Phi^*\Phi) = 0$, and the action is given by the formula (6). From the corresponding Euler-Lagrange equations of motion

$$\Box\Phi(t,\varphi) \;=\; m^2\Phi(t,\varphi) \;, \tag{12}$$

it folows that $a_k(t) = a_k(0)e^{\pm i\omega_k t}$, with $\omega_k = \sqrt{k^2 + m^2}$, $k \in \mathbf{Z}$. We see that the free field on C represents a system of independent oscillators with frequencies ω_k.

The dynamics of the model is described by the propagator (fundamental solution) of the equation:

$$(\Box - m^2)G_0(t,x;t',x') \;=\; \delta(t,x;t',x') \;, \tag{13}$$

where on r.h.s. it appears the δ-function on a cylinder

$$\delta(t,x;t',x') \;=\; \frac{1}{2\pi}\delta(t-t')\sum_{k\in\mathbf{Z}}(x_+x'_-/\rho^2)^k \;. \tag{14}$$

It can be straightforwardly shown that

$$G_0(t,x;t',x') \;=\; \frac{1}{2\pi}\int_{\mathbf{R}} d\omega \;\sum_{k\in\mathbf{Z}} \frac{e^{i\omega(t-t')}}{\omega^2 - \omega_k^2 + i\varepsilon}(x_+x'_-/\rho^2)^k \;. \tag{15}$$

In the energy-momentum respresentation the propagator is given as

$$\tilde{G}_0(\omega,k) \;=\; \frac{1}{\omega^2 - \omega_k^2 + i\varepsilon} \;. \tag{16}$$

The tadpole $\Gamma_0 = G_0(t,x;t,x)$ is given by a logarithmically divergent integral

$$\Gamma_0 \;=\; \frac{1}{2\pi}\int_{\mathbf{R}} d\omega \sum_{k\in\mathbf{Z}} \frac{1}{\omega^2-\omega_k^2+i\varepsilon} \;=\; \sum_{k\in\mathbf{Z}} \frac{1}{\omega_k} \;=\; \infty\,. \tag{17}$$

This leads to UV-divergencies of Feynman diagrams appearing in a perturbative QFT expansion of the model described by the action (5).

Note: The commutative algebra $\mathcal{A}_0$ is defined as the algebra of polynomials in the variables $t, x_\pm$ satisfying the defining relations

$$[t,x_\pm] \;=\; 0\;,\; [x_+,x_-] \;=\; 0\;,$$
$$x_+x_- \;=\; x_-x_+ \;=\; \rho^2\;, \rho>0\,. \tag{18}$$

The multiplication by polynomials from $\mathcal{A}_0$ is not well-defined for any field from the Hilbert space $\mathcal{H}$ satisfying the condition

$$I_0[\Phi^*\Phi] \;<\; \infty\,.$$

We have to reduce $\mathcal{H}$ to the subspace $\mathcal{S}$ of fields of the form (2) with properly restricted coefficient functions $a_k(t)$:

$$k^l t^m \partial_t^n a_k(t) \tag{19}$$

should be uniformly bounded in $t\in\mathbf{R}$ for any natural l,m,n. The multiplication by polynomials from $\mathcal{A}_0$ maps $\mathcal{S}\to\mathcal{S}$. The algebra $\mathcal{A}_0$ fully characterizes the cylinder C as a manifold.

3 The noncommutative case

We start with the 2D Euclidean Lie algebra $e(2)$ with generators $X_0, X_\pm$, satisfying the relations (7). This algebra possesses one series of infinitedimensional unitary representations π_ρ (parametrized by a real parameter $\rho>0$) which can be realized in the Hilbert space $L^2(S^1)$. In this representation

$$X_0 \;=\; -i\partial_\varphi\;,\; X_\pm \;=\; \rho e^{\pm i\varphi}\,. \tag{20}$$

The Casimir operator is X_+X_-. In the representation π_ρ takes the value $X_+X_- = \rho^2$ (for simplicity we use the notation X_μ instead of $\pi_\rho(X_\mu)$, $\mu=0,\pm$).

The algebra $\mathcal{A}$, a deformation of the algebra $\mathcal{A}_0$, is defined as the algebra of polynomials in $t=\lambda X_0$ and $x_\pm = -X_\mp$ satisfying the defining relations

$$[t,x_\pm] \;=\; \pm\lambda x_\pm\;,\; [x_+,x_-] \;=\; 0\;,$$
$$x_+x_- \;=\; x_-x_+ \;=\; \rho^2\;,\; \rho>0\,. \tag{21}$$

The fields are now functions in the noncommutative variables $t, x_\pm$ possessing the expansion

$$\Phi(t,x) \;=\; a_0(t) \;+\; \sum_{k=1}^{\infty} a_k(t)x_+^k \;+\; \sum_{k=1}^{\infty} a_{-k}(t)x_-^k\,. \tag{22}$$

The integral on a noncommutative cylinder we define as

$$I_\lambda \;=\; \lambda \mathrm{Tr}[\Phi] \;=\; \lambda \sum_{n\in\mathbf{Z}} a_0(\lambda n) \; . \tag{23}$$

We see that the standard measure dt on $\mathbf{R}$ is replaced by a spectral meausure of t: the uniform step-measure with the height λ at points λn, $n \in \mathbf{Z}$.

Note: The multiplication by elements from $\mathcal{A}$ is not well-defined for any field from the Hilbert space $\mathcal{H}$ satisfying the condition

$$I_\lambda[\Phi^*\Phi] \;<\; \infty \; .$$

However, $\mathcal{A}$ acts in the subspace $\mathcal{S}$ of fields of the form (22) with operators $a_k(t)$ satisfying the spectral condition:

$$k^l n^m a_k(\lambda n) \tag{24}$$

is bounded uniformly in k for any natural $l, m, |n|$. The multiplication by polynomials from $\mathcal{A}$ maps $\mathcal{S} \to \mathcal{S}$. The noncommutative algebra $\mathcal{A}$ defines the noncommutative cylinder C.

In the commutative case the Fourier transform of $a_k(t)$ is the function

$$\tilde{a}_k(\omega) \;=\; \frac{1}{2\pi} \int_{\mathbf{R}} dt\, e^{i\omega t} a_k(t) \;,\; \omega \in \mathbf{R} \; . \tag{25}$$

In the noncommutative case the corresponding spectral Fourier transform of the operator $a_k(t)$ is

$$\tilde{a}_k(\omega) \;=\; \frac{1}{2\pi} \sum_{n\in\mathbf{Z}} e^{i\omega\lambda n} a_k(\lambda n) \; . \tag{26}$$

We see that $\tilde{a}_k(\omega)$ is periodic on $\mathbf{R}$ with the period $2\pi/\lambda$.

The field action we define in the noncommutative case analogously as in the commutative version by

$$S[\Phi, \Phi^*] \;=\; I_\lambda[:\Phi^*(\Box - m^2)\Phi \;-\; V(\Phi^*\Phi) :] \; , \tag{27}$$

replacing the integration on a commutative cylinder by its noncommutative analog. Moreover, the wave operator is now given as in (11) but with Poisson brackets replaced by commutators according to then rule $\{.,.\} \to \frac{i}{\rho\lambda}[.,.]$. Thus, in the noncommutative case

$$\Box\Phi \;=\; -\frac{1}{\rho^2\lambda^2}\{[x_0,[x_0,\Phi]] \;-\; [x_+,[x_-,\Phi]]\} \; . \tag{28}$$

The integrand is an operator and an ordering prescription $: \ldots :$ is needed. It is defined as follows:

- we express the field Φ in terms of the Fourier expansion (22) and calculate the Lagrangian density,

- there appear the coefficients $a_k(t), a_k(t \pm \lambda)$ and $a_k^*(t), a_k^*(t \pm \lambda)$, and we collect them as a right (or equivalently left) factor in each term.

Let us restrict ourselves to the free field case. In this case the action is given as

$$S_0[\Phi,\Phi^*] \;=\; \lambda \sum_{n\in\mathbf{Z}} \sum_{k\in\mathbf{Z}} \{ \frac{1}{\lambda^2}[a_k^*(n\lambda+\lambda) - a_k^*(n\lambda)][a_k(n\lambda+\lambda) - a_k(n\lambda)]$$

$$- (k^2+m^2) a_k^*(n\lambda) a_k(n\lambda)\} \; . \tag{29}$$

The corresponding Euler-Lagrange equations of motion are obtained in a standard way by performing the variations

$$a_k^*(n\lambda) \;\to\; a_k^*(n\lambda) \;+\; \varepsilon_k^*(n\lambda) \; , \; a_k(n\lambda) \;\to\; a_k(n\lambda) \;+\; \varepsilon_k(n\lambda) \; . \tag{30}$$

The first variation leads to the equation of motion

$$\frac{1}{\lambda^2}[a_k(n\lambda+\lambda) - 2a_k(n\lambda) + a_k(n\lambda-\lambda)] \;=\; (k^2+m^2) a_k(n\lambda)\} \; , \tag{31}$$

and the second one to the conjugated equation. The model thus describes a system of independent oscillators subjected to a discrete time evolution.

Performing the inverse Fourier transformation

$$a_k(n\lambda) \;=\; \frac{\lambda}{2\pi} \int_{-\pi/\lambda}^{+\pi/\lambda} d\omega \; e^{i\omega n\lambda} \tilde{a}_k(\omega) \; , \tag{32}$$

we diagonalize the free field action. In fact, using the well-known formula

$$\sum_{n\in\mathbf{Z}} e^{i(\omega-\omega')n\lambda} \;=\; \frac{2\pi}{\lambda} \delta(\omega-\omega') \; , \tag{33}$$

we obtain the free field action in the form

$$S_0[\Phi,\Phi^*] \;=\; \frac{1}{2\pi} \int_{-\pi/\lambda}^{+\pi/\lambda} d\omega \; \sum_{k\in\mathbf{Z}} \tilde{a}_k^*(\omega)[\Omega_\lambda^2(\omega) - (k^2+m^2)]\tilde{a}_k(\omega) \; , \tag{34}$$

where $\Omega_\lambda(\omega) = (2/\lambda)\sin(\omega\lambda/2)$. The free field equations following from (34) have the form

$$\Omega_\lambda^2(\omega) \;=\; k^2+m^2 \; . \tag{35}$$

Since $|\Omega_\lambda^2(\omega)| \leq 2\lambda^{-1}$, the set of modes is restricted, and the admissible values of k are given as

$$k \;-\; \text{integer} \;\; , \;\; |k| \leq \sqrt{4\lambda^{-2} - m^2} \; . \tag{36}$$

We stress that the consistency requires that the mass m should be less than the maximal "Planck" mass $2\lambda^{-1}$.

We see that the spectrum of modes is restricted, and one can expect that this can lead to a removal of UV-divergencies from Feynman diagrams. It was proved in [14] that this is indeed true. It was shown that the proper formula for the propagator

$$G_0(t,x;t'x') \;=\; \frac{1}{2\pi} \int_{-\pi/\lambda}^{+\pi/\lambda} d\omega \; \sum_{k\in\mathbf{Z}} \frac{e^{i\omega\lambda(n-n')}}{\Omega_\lambda^2(\omega) - k^2 - m^2 + i\varepsilon} (x_+ x'_-/\rho^2)^k \; , \tag{37}$$

leads to a convergent tadpole integral. Moreover, it was proved that the vertex modifications, induced by the noncommutativity of the space-time, do not spoil the finitness the tadpole contribution. This means that there are no UV-divergencies in Feynman diagrams entering the perturbative expansion.

Note: If the field is quantized the operators $a_k(t)$ and $a_k^*(t)$ are either annihilation or creation in the usual Fock space. This leads to an additional Fock ordering. The second part of the ordering rule presented above should be then replaced by:

- those operators from $a_k(t), a_k(t \pm \lambda), a_k^*(t), a_k^*(t \pm \lambda)$ which are annihilation (creation) we collect as a right- (left-) factor in each term of a Lagrangian density.

4 Concluding remarks

We described a field theoretical model on a noncommutative cylinder. This is one of the first models formulated on a noncommutative analog of a noncompact space-time.

The manifold in question possesses a well defined notion of a time, and moreover for a fixed time the corresponding space slice is the *compact* circle S^1. The noncommutativity (introduced by the elementary length λ) induces

- the *discrete* time evolution, and
- the *regularization* - a supression of high frequency modes.

We explicitely demonstrated the last point for a scalar free field only, its validity was proved for a self-interacting scalar field in [14]. One intuitively feels that this will be a typical feature for more general models on a noncommutative cylinder, e.g. possessing various interacting fields (scalar, spinor, gauge, etc.). It would be desirable to extend this approach

- to relativistic models on a noncommutative causal hyperboloid, and
- to higher dimensional, i.e. 4D, models.

All these investigations are under study. We hope that we shall be able to discuss more physical models in the next future.

Acknowledgement One of the authors (P.P.) would like to thank to his friend and former teacher Jan Pišút (Comenius University, Bratislava) who had introduced him some thirty years ago to the area of theoretical physics and from that time supported him with fruitful advices and critical remarks.

This work was supported by the Fond zur Förderung der Wissenschaftlichen Forschung in Österreich project P11783-Phy and by the Slovak Grant Agency VEGA project 1/4305/97

References

[1] A. Connes: *Publ. IHES* **62** (1986) 257.

[2] A. Connes: *Geometrie Noncommutative* (Inter Editions, Paris 1990).

[3] A. Connes: *Gravity coupled with matter and the foundation of non commutative geometry*, hep-th/9603053.

[4] M. Dubois-Violette: *C. R. Acad. Sci. Paris* **307**, Ser. I (1988) 403.

[5] M. Dubois-Violette, R. Kerner, J. Madore: *Journ. Math. Phys.* **31** (1990) 316.

[6] F. A. Berezin: *Commun. Math. Phys.* **40** (1975) 153.

[7] J. Hoppe: *Elem. Part. Res. J.* **80** (1989) 145 .

[8] J. Madore: *Journ. Math. Phys.* **32** (1991) 332.

[9] H. Grosse, P. Prešnajder: *Lett. Math. Phys.* **28** (1993) 239.

[10] H. Grosse, C. Klimčík, P. Prešnajder: *Int. Journ. Theor. Phys.* **35** (1996) 231.

[11] H. Grosse, C. Klimčík, P. Prešnajder: *Commun. Math. Phys.* **178** (1996) 507.

[12] H. Grosse, C. Klimčík, P. Prešnajder: *Commun. Math. Phys.* **185** (1997) 155.

[13] H. Grosse, P. Prešnajder: *A noncommutative regularization of the Schwinger model*, Vienna University preprint UWThPh-17-1998, to be published in Lett. Math. Phys. (1998).

[14] M. Chaichian, A. Demichev, P. Prešnajder: *Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences*, Helsinki University preprint HIP-1998-77/TH, hep-th/981280.