**acta physica slovaca vol. 49 No. 2, 145 – 150 April 1999**

# BE CONDENSATION AND INDEPENDENT EMISSION: STATISTICAL PHYSICS INTERPRETATION*

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Received 15 July 1998, accepted 13 January 1999

Recent results on effects of Bose-Einstein symmetrization in a system of independently produced particles are interpreted in terms of statistical physics. For a large class of distributions, the effective sizes of the system in momentum and in configuration space are shown to shrink when quantum interference is taken into account.

PACS: 13.60.2e, 13.85.Ni, 25.75.Gz

**1.** In recent papers [1, 2] we worked out the implications of the assumption that the identical pions generated in multiple production processes have only the correlations due to Bose-Einstein statistics. This was taken to mean that observable distributions can be evaluated in two steps. First, for the unphysical case of distinguishable pions independence is assumed, i.e. the poissonian multiplicity distribution

$$P^{(0)}(N) = e^{-\nu}\frac{\nu^N}{N!} \tag{1}$$

and the density matrix for each multiplicity in the form of a product of single particle density matrices. In momentum representation, for $N$ particles we have

$$\rho_N^{(0)}(q, q') = \prod_{i=1}^{N} \rho^{(0)}(q_i, q_i'), \tag{2}$$

where $q_i$ denotes the momentum vector of particle $i$ and $q$ is the set of all $N$ momenta[3]. At this stage the momentum distribution is given by the diagonal elements of the density matrix

$$\Omega_N^{(0)}(q) = \rho_N^{(0)}(q, q) \tag{3}$$

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[3] The special case, when $\rho^{(0)}$ is a gaussian, has been studied already by a number of authors [2]-[7]

and is normalized to unity

$$\int dq\Omega_N^{(0)}(q) = 1. \tag{4}$$

In the second step the distribution is symmetrized, i.e. the resulting momentum distribution is obtained according to the prescription

$$\Omega_N(q) = \frac{1}{N!}\sum_{P,P'}\rho^{(0)}(q_P, q'_P) = \sum_P Re\rho^{(0)}(q, q_P), \tag{5}$$

where the first sum runs over all permutations $P, P'$ of the momenta. As seen from (5), the momentum distribution $\Omega_N(q)$ is not normalized to one any more. Consequently, the multiplicity distribution changes from (1) into

$$P(N) = C\frac{\nu^N}{N!}\int \Omega_N(q)dq \tag{6}$$

where $C$ is a normalization constant.

We found for the generating function of the multiplicity distribution [1] and for the generating functional for the multiparticle correlation functions in momentum space [2] explicit expressions in terms of the eigenfunctions and eigenvalues of the single particle density matrix $\rho^{(0)}$. In the present paper we describe a reinterpretation of this model and show how the results can be simply obtained by using standard statistical physics. This explains directly the physical meaning of the results obtained in [1, 2].

**2.** The density operator corresponding to the density matrix $\rho^{(0)}(q_i, q'_i)$ can be expressed in terms of its eigenvectors and eigenvalues

$$\hat{\rho}^{(0)} = \sum_n |n > \lambda_n < n|. \tag{7}$$

Our first remark is that this operator corresponds to a single particle canonical distribution, if we make the identification[4]

$$\lambda_n = \frac{1}{Z}e^{-\beta\epsilon_n}, \tag{8}$$

where $\beta = \frac{1}{kT}$, $\epsilon_n$ is the energy ascribed to the state $n$, and $Z$ is the normalizing factor which ensures that the density operator has the trace equal one

$$\sum_n \lambda_n = 1. \tag{9}$$

In terms of statistical physics, $Z$ is the single particle partition function:

$$Z = \sum_n e^{-\beta\epsilon_n}. \tag{10}$$

[4]We consider only states with $\lambda_n > 0$. The states with $\lambda_n = 0$ correspond to infinite energy and do not play any role in our argument.

The sum is over all states of the particle, thus a given energy $\epsilon_n$ can occur more than once. The hamiltonian with eigenstates $|n>$ and eigenvalues $\epsilon_n$ is, of course,

$$H = \sum_n |n > \epsilon_n < n|. \tag{11}$$

For instance, from results of [1] and [2] we find that for the Gaussian single particle density matrix in one dimension

$$\rho^{(0)}(q, q') = \frac{1}{\sqrt{2\pi\Delta^2}} \exp -\frac{q_+^2}{2\Delta^2} - \frac{1}{2}R^2 q_-^2, \tag{12}$$

where $\Delta^2$ and $R^2$ are positive constants constrained by the condition $R\Delta \geq 1/2$ following from Heisenberg's uncertainty relation, the Hamiltonian is that of a harmonic oscillator with

$$m\omega = \hbar \left(\frac{R}{\Delta}\right)^2 \tag{13}$$

and

$$\hbar\omega = -kT \ln \frac{2R\Delta - 1}{2R\Delta + 1}. \tag{14}$$

Inversely, for any Hamiltonian with a known set of eigenfunctions $\psi_n(q)$ and eigenvalues $\epsilon_n$ one can construct the corresponding density matrix $\rho^{(0)}(q, q')$ using the formulae (7) and (8). This allows to obtain a fairly large class of explicitly solvable models.

Let us consider now a set of $N$ indistinguishable particles, where the Hamiltonian is the sum over all the particles of single particle Hamiltonians (11). We assume that all these particles have been produced independently[5] and that the probability of producing $N$ particles is $\bar{\nu}^N$. Let us consider first the subset of particles in state $|n>$. Since the number of particles in this subset is not fixed, we use the grand-canonical ensemble. The probability of finding exactly $N$ particles in state $|n>$ is

$$P_n(N) = \frac{1}{\mathcal{Z}_n} \bar{\nu}^N e^{-\beta N \epsilon_n}. \tag{15}$$

The parameter $\bar{\nu}$, known as fugacity, is related to the chemical potential $\mu$ by the formula

$$\bar{\nu} = e^{\beta\mu} \tag{16}$$

and to $\nu$ by $\bar{\nu} = \nu/Z$. The normalizing factor $\mathcal{Z}_n$, known as the grand partition function, ensures that the sum of all the probabilities equals one:

$$\mathcal{Z}_n = \sum_{N=0}^{\infty} \left(\bar{\nu} e^{-\beta\epsilon_n}\right)^N = \frac{1}{1 - \bar{\nu} e^{-\beta\epsilon_n}}. \tag{17}$$

[5] These assumptions correspond to the mean field approximation in statistical physics. Thermodynamic equilibrium is not assumed. For a discussion assuming thermodynamic equlibrium cf. [8].

This formula makes sense only if the geometrical series is convergent, i.e. for $\mu < \epsilon_n$. The grand partition function for the whole system is a product of the grand partition functions for the independent subsystems, thus it is

$$\mathcal{Z} = \prod_n \frac{1}{1 - \bar{\nu} e^{-\beta\epsilon_n}}. \tag{18}$$

The grand partition function contains much information about the system. In particular it can replace the generating function for calculation of the multiplicity distributions. For instance, the average number of particles in state $n$ is

$$-kT \frac{\partial(\ln \mathcal{Z}_n)}{\partial\mu} = \frac{1}{e^{\beta(\epsilon_n - \mu)} - 1} = \frac{\nu\lambda_n}{1 - \nu\lambda_n}. \tag{19}$$

The first form is the well-known Bose-Einstein distribution, the second is the result obtained in [1]. Thus the symmetrization according to formula (5) corresponds to the replacement of the Boltzmann multiplicity distribution by the Bose-Einstein one. For the probability of no particle in state $n$ ($N_n = 0$) we get from (15)

$$P_n(0) = \frac{1}{\mathcal{Z}_n} = 1 - \bar{\nu} e^{-\beta\epsilon_n} = \frac{1}{\bar{N}_n + 1}. \tag{20}$$

Two limiting cases are of interest here. When all the occupation numbers $\bar{N}_i$ are very small $P(0) = \prod P_n(0) \approx e^{-\sum \bar{N}_n} = e^{-\bar{N}}$ is very small for large $\bar{N}$. When almost all the particles are in the ground state (BE condensation), $P(0) \approx (1 + \bar{N})^{-1}$ and this probability becomes much larger. Thus the probability of producing an event with no $\pi^0$'s (centauro event [9]) is greatly enhanced by symmetrization, when a significant fraction of particles is in the condensate. This happens when $\mu \to \epsilon_0$ from below, or in terms of the eigenvalues of the density matrix when $\nu\lambda_0 \to 1$ from below.

**3.** The momentum distribution and correlation functions can be described along the same lines. It was shown in [2] that they can all be expressed in terms of one function of two variables $L(q, q')$ given by

$$L(q, q') = \sum_n \psi_n(q)\psi_n^*(q') \frac{\nu\lambda_n}{1 - \nu\lambda_n}. \tag{21}$$

In particular, we have

$$\omega(q) = L(q, q) \tag{22}$$

for the inclusive single particle distribution, and

$$K_2(q_1, q_2) = |L(q_1, q_2)|^2 \tag{23}$$

for the two-particle correlation function.

Using (8) and (16), Eq.(21) can rewritten as

$$L(q, q') = \sum_n \psi_n(q)\psi_n^*(q') \frac{1}{e^{\beta(\epsilon_n - \mu)} - 1} \tag{24}$$

Thus, it is just the element of the density matrix for the pure state $n$ averaged over the Bose-Einstein distribution. Replacing the Bose-Einstein weights by the Boltzmann weights (8) one recovers the unsymmetrized single particle density matrix $\rho^{(0)}(q,q')$. At this point let us observe that the unsymmetrized density matrix $\rho^{(0)}(q,q')$ corresponds better than the symmetrized one to the physical intuition since it contains the parameters which are easier to intepret. This rises an interesting question, to what extent the apparent parameters of the system, as determined from measured $\omega(q)$ and $K_2(q,q')$, are modified with respect to the original "physical" parameters given by $\rho^{(0)}(q,q')$.

This question was explicitly solved for the case of the Gaussian density matrix (corresponding to the hamiltonian of a harmonic oscillator): the resulting distributions are narrower (both in momentum and in coordinate space) than the original, unsymmetrized distributions [2, 5, 6, 7, 10]. It is not obvious, however, how general this result is. Below we discuss this problem for the case of hamiltonians of the form

$$H = \frac{p^2}{2m} + V(x). \tag{25}$$

with $V(x) \to +\infty$ when $|x| \to \infty$, to ensure the convergence of the trace of the density matrix.

To this end let us first observe that the Bose-Einstein weights, when compared with those of Boltzmann, enhance the contribution from states with lower energy in the sum (24). Thus, if the wave functions $\psi_n(x)$ and $\psi_n(q)$ broaden with increasing energies $\epsilon_n$ (decreasing $\lambda_n$), the symmetrized distributions are narrower than the unsymmetrized ones, i.e. we recover the qualitative result obtained for the Gaussian density matrix. For such a situation to occur, it is sufficient for example that the potential $V(x) = \lambda|x|^\alpha$ where $\lambda$ and $\alpha$ are positive constants[6]. Unfortunately, the necessary condition(s) are not so easy to determine. It is also not immediately obvious, what this condition implies for the shape of the density matrix $\rho^{(0)}(q,q')$.

This argument also holds for the distribution in configuration space, as determined from the correlation function $K_2(q_1,q_2)$. In this case we argue that the distribution in the variable $q_1 - q_2$ broadens when the symmetrization effects are introduced. To this end let us observe that if all eigenvalues $\lambda_n$ were the same, the sums in (21) and (24) would be proportional to $\delta(q-q')$. Thus $L(q_1,q_2)$ would be infinitely narrow in $q_1 - q_2$, corresponding to an infinite volume for particle production. Introducing the Boltzmann weights (8) cuts the contribution from large energy levels and enhances the lower ones. The result is broadening of $L$. Since, as we argued above, the Bose-Einstein weights work even stronger in the same direction, they will broaden the distribution even more.

We thus conclude that the narrowing of the momentum and configuration space distributions as a consequence of the symmetrization of the multiparticle wave functions (first observed for the Gaussian density matrix) is actually a much more general phenomenon, valid for a rather broad class of distributions. More work is needed, however, to determine precisely the physical conditions which determine the behaviour of the effective widths of the distributions after symmetrization.

**4.** For fermions the difference is that $P_n(N) = 0$ for $N > 1$. Thus the formulae (17) and

[6]The authors thank V.Zakharov for calling their attention to the virial theorem, which implies that.

(18) are replaced by

$$\mathcal{Z}_n^{(F)} = 1 + \bar{\nu} e^{-\beta\epsilon_n}; \quad \mathcal{Z}^{(F)} = \prod_n (1 + \bar{\nu} e^{-\beta\epsilon_n}) \tag{26}$$

The formula for the average population of state $n$ becomes

$$-kT \frac{\partial \ln \mathcal{Z}_n}{\partial \mu} = \frac{1}{e^{\beta(\epsilon_n - \mu)} + 1} = \frac{\nu\lambda_n}{1 + \nu\lambda_n} \tag{27}$$

and, of course, there is no condensation. After symmetrization the elements of the density matrix $\rho^{(0)}$ are replaced by

$$L(q, q') = \sum_n \psi_n(q)\psi_n^*(q') \frac{1}{e^{\beta(\epsilon_n - \mu)} + 1} \tag{28}$$

One sees from (28) that the Fermi-Dirac weights reduce the contribution from the low-energy states as compared to the Boltzmann weights which would be there without (anti) symmetrization of the wave function. Thus the argument given in the previous section implies that (anti) symmetrization of the wave function leads to broadening of the fermion distribution both in momentum and configuration space.

**5.** In conclusion, we have shown that the assumption that the identical bosons (fermions) generated in multiple production processes have only the correlations due to Bose-Einstein (Fermi-Dirac) statistics can be reformulated in the language of statistical physics. It corresponds to the standard Bose-Einstein (Fermi-Dirac) distribution of particles in the mean field approximation. Using this formulation we have shown for a large class of distributions that symmetrization of the multiparticle wave function implies narrowing of the particle spectra both in momentum and in configuration space (broadening is expected in case of fermions). This generalizes the result found earlier for the gaussian distributions [2]-[7].

**Acknowledgements** We would like to thank Piotr Bialas, Ján Pišút and Andrzej Staruszkiewicz for very useful discussions, and to Ulrich Heinz for an interesting correspondence. This work was supported by the KBN Grant No 2 P03B 086 14.

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