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THE LPM EFFECT: WHY THERE IS NO COHERENCE ON THE COHERENCE LENGTH?*

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The noncoherence of radiation of a high-energy particle in a medium, down to frequencies at which the coherence length exceeds the mean free path of the particle considerably, is discussed. The noncoherence can be assigned to an interplay between destructive and constructive interference of the radiation coming from separate collisions. In this explanation crucial role is played by the specific form of the law governing the random walk of the particle in a plane perpendicular to initial velocity.

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1 Introduction

Landau and Pomeranchuk observed in 1953 that if a high-energy particle moves through a medium, its radiation is damped at low frequencies due to the fact that contributions from individual scatterings sum up coherently [1]. A formula for the energy of radiation valid at arbitrary frequencies was obtained, both in classical and in quantum electrodynamics, by Migdal [2]. After the original papers on Landau-Pomeranchuk-Migdal (LPM) effect appeared, the story seemed to be more or less complete. Recently, however, a new chapter is being written of it. An experiment has been performed on LPM effect at SLAC [3] and, what is perhaps more important yet, the effect acquired a new field of application in the physics of quark-gluon plasma [4]. As a result the theory has been reformulated and extended [5].

Since the damping of radiation in LPM effect is ascribed to coherence, it is natural to ask what is the role played by the coherence length in it. One would expect that LPM effect takes place if the particle undergoes on the average more than one collision within one coherence length. This is, however, not the case. The damping starts at a frequency at which the coherence length is typically much greater than the mean free path of the particle. Thus the question arises where the noncoherence at higher frequencies comes from.

An effective tool for the qualitative discussion of LPM effect is an approximation of the amplitude of radiation called factorization limit [6]. It was, in particular, implicitly used in an elementary derivation of the low-frequency spectrum in [7]. Here I will adopt the factorization

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limit in a search for an answer to the question stated above. In Section II the coherence length is defined, in Section III the original argument leading to LPM effect is recalled, in Section IV the factorization limit is introduced, in Section V an explanation of noncoherence above the critical frequency is given, in Section VI the spectrum of radiation is computed and in Section VII two concluding remarks are made.

2 Definition of coherence length

The quantity named by Landau and Pomeranchuk coherence length will be called formation length here, as it is common in present literature. By introducing it the line of reasoning of [8] will be followed.

The energy radiated by a charged particle per unit interval of frequencies and per unit solid angle is

$$\frac{d\epsilon}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2}\left|\int_{-\infty}^{\infty}\mathbf{n}\times\mathbf{v}e^{i\omega(t-\mathbf{n}.\mathbf{r})}dt\right|^2, \tag{1}$$

where q is the charge of the particle, ω is the frequency of radiation, $\mathbf{n}$ is the unit vector directed to the observer, $\mathbf{r}$ is the radius vector of the particle, $\mathbf{v}$ is the velocity of the particle and t is time. (The system of units is used in which $c = 1$.) Suppose that the particle undergoes a multiple scattering. The contribution to the integral in Eq. (1) from the ith segment of the trajectory of the particle is proportional to the expression

$$\frac{1}{i\omega(1-\mathbf{n}.\mathbf{v}_i)}\left[e^{i\omega\tau_i(1-\mathbf{n}.\mathbf{v}_i)}-1\right],$$

where $\mathbf{v}_i$ is the velocity of the particle on the ith segment of trajectory and τ_i is the time spent by the particle there. If the time τ_i is much greater than $1/[\omega(1-\mathbf{n}.\mathbf{v}_i)]$, the oscillating first term in square brackets can be dropped and the whole expression is of order $1/\omega$. If, on the other hand, the time τ_i is much less than $1/[\omega(1-\mathbf{n}.\mathbf{v}_i)]$, the expression in square brackets is proportional to $\omega\tau_i$ and the whole expression reduces to $\tau_i \ll 1/\omega$. In the former case the contribution from the ith segment to the total energy of radiation is of the same order of magnitude as the contributions from the in- and outgoing particle (obtained by setting $\tau_{in} = \tau_{out} = \infty$), while in the latter case the contribution from the ith segment is damped and the segment can be ignored. In fact, one can ignore all segments of trajectory passed by the particle during the time $1/[\omega(1-\mathbf{n}.\mathbf{v})]$, where $\mathbf{v}$ is the initial velocity of the particle, provided the velocity of the particle is not influenced by collisions considerably. The corresponding distance is called the coherence, or formation, length. According to this definition, the formation length is

$$l_f = \frac{v}{\omega(1-\mathbf{n}.\mathbf{v})}. \tag{2}$$

This expression is of order v/ω, except when the particle is ultrarelativistic and the angle between the vectors $\mathbf{n}$ and $\mathbf{v}$ is small enough. Define the cone of emission as a narrow cone around the direction of vector $\mathbf{v}$ with the angle at the peak equal to γ^{-1}, where $\gamma = (1-v^2)^{-1/2}$. Such a cone contains practically all the energy radiated within the formation length, hence the name.

If the particle is ultrarelativistic and vector $\mathbf{n}$ points into the cone of emission, the formation length is

$$l_f \sim \frac{\gamma^2}{\omega}. \tag{3}$$

In an argument leading to formation length the limit $\omega \to 0$ can be used [8]. However, there is no need for that. The value of ω is arbitrary, as long as we restrict ourselves to classical electrodynamics. Of course, in order to guarantee the validity of classical electrodynamics one has to require $\hbar\omega \ll E$, where E is the energy of the particle.

3 The argument by Landau and Pomeranchuk

One may try to understand the nature of LPM effect by looking into the original paper by Landau and Pomeranchuk (the first paper of ref. [1]). Let us recall the argument presented there.

Consider multiple scattering starting from the point $\mathbf{r} = 0$ at the moment $t = 0$. The phase of the integrand on the right hand side of Eq. (1) at the moment t is

$$\Phi = \omega \int_0^t (1 - \mathbf{n}.\mathbf{v}_\xi) d\xi, \tag{4}$$

where $\mathbf{v}_\xi$ is the velocity of the particle at the moment ξ. If the particle moves ultrarelativistically and vector $\mathbf{n}$ has the direction of initial velocity, it holds

$$1 - \mathbf{n}.\mathbf{v}_\xi \doteq \frac{1}{2}[\gamma^{-2} + \alpha^2(\xi)],$$

where $\alpha(\xi)$ is the total scattering angle at the moment ξ. The angle α changes with time by the "walk of drunken sailor". Consequently, its mean square at the moment t is

$$\overline{\alpha^2} = \frac{1}{l} < \alpha^2_{ind} > t,$$

where l is the mean free path of the particle, α_{ind} is the scattering angle acquired by the particle at one scattering centre (the subscript "ind" stays for "individual scattering"), overline denotes an average over trajectories and angle brackets denote an average over individual scatterings. As a result, one finds

$$\overline{\Phi} = \frac{1}{2}\omega(\gamma^{-2} + \frac{1}{2l} < \alpha^2_{ind} > t)t. \tag{5}$$

Suppose that t is the time during which the particle passes the formation length, $t \doteq l_f$. Using Eq. (3) one finds that the two terms in the brackets coincide by the order of magnitude at the frequency

$$\omega_{crit} = \frac{1}{l}\gamma^4 < \alpha^2_{ind} > . \tag{6}$$

At higher frequencies the second term is negligible, while at lower frequencies it is dominant. Thus, Landau and Pomeranchuk argue, in the former case the multiple scattering does not play any role and the Bethe-Heitler formula for noncoherent radiation is valid, while in the latter case the multiple scattering must be taken into account and the Bethe-Heitler formula must be modified.

If one scrutinizes the argument by Landau and Pomeranchuk in the light of what has been said about the meaning of the formation length, one finds it not too compelling. No matter whether collisions influence the phase Φ or not, if the formation length exceeds the mean free path of the particle, destructive interference of the radiation coming from neighbouring collisions necessarily takes place. That means that radiation is coherent and there is no *a priori* reason to expect that the Bethe-Heitler formula will work. Let us show that this is what really happens. It holds $< \alpha_{ind}^2 > \propto \gamma^{-2}$ and, consequently, $\omega_{crit} = < \alpha_{ind}^2 >_0 \gamma^2/l$, where $< \alpha_{ind}^2 >_0$ is the value of $< \alpha_{ind}^2 >$ at nonrelativistic velocities. Inserting this into Eq. (3) one obtains

$$l_{f,crit} = \frac{l}{< \alpha_{ind}^2 >_0}.$$

Since $< \alpha_{ind}^2 >_0$ is typically much less then 1, the formation length at critical frequency is typically much greater than the mean free path of the particle. Thus the radiation is coherent in a large interval of frequencies higher than critical.

4 Factorization limit

Consider a charged particle scattering at the moment $t = 0$ on an isolated centre located at the point $\mathbf{r} = 0$. If the velocity of the particle changes from $\mathbf{v}$ to $\mathbf{v}'$ during the scattering, the integral in Eq. (1), with the oscillating terms coming from lower and upper limit of the integration region dropped, is

$$\mathbf{A} = \frac{1}{i\omega}(\mathbf{a} - \mathbf{a}'), \tag{7}$$

where

$$\mathbf{a} = \frac{\mathbf{n} \times \mathbf{v}}{1 - \mathbf{n}.\mathbf{v}} \tag{8}$$

and $\mathbf{a}'$ is the same expression with $\mathbf{v}$ replaced by $\mathbf{v}'$. Note that the oscillating terms are an artifact of the way the Maxwell theory is treated, not a product of the theory itself. They come from an illegal integration per partes performed when deriving Eq. (1).

Let us now pass to the multiple scattering. Suppose that the particle scatters subsequently on N_{tot} centers located at the points $\mathbf{r}_i$, the scatterings take place at the moments t_i, and the velocity of the particle between the ith and $(i+1)$th scattering is $\mathbf{v}_i$. The integral in Eq. (1) is

$$\mathbf{A} = \sum_{i=1}^{N_{tot}} \mathbf{A}_i, \tag{9}$$

where

$$\mathbf{A}_i = \frac{1}{i\omega}(\mathbf{a}_{i+1} - \mathbf{a}_i)e^{i\Phi_i}, \tag{10}$$

$\mathbf{a}_i$ is the expression (8) with $\mathbf{v}$ replaced by $\mathbf{v}_i$ and $\Phi_i = \omega(t_i - \mathbf{n}.\mathbf{r}_i) = \Phi$ of the previous section evaluated at the ith collision. Choose $\mathbf{r}_1 = t_1 = 0$. If the velocity does not change considerably due to collisions, it holds $\mathbf{r}_i \doteq \mathbf{v}t_i$, where $\mathbf{v} = \mathbf{v}_1$. and the phase of $\mathbf{A}_i$ is

$$\Phi_i \doteq \omega t_i(1 - \mathbf{n}.\mathbf{v}).$$

Let N be the ordering number of the scattering at which the phase reaches the value 1. The path passed by the particle can be divided into three parts: part 1 on which $i \ll N$, $\Phi_i \doteq 0$, part 2 on which $i \approx N$, $\Phi_i \approx 1$, and part 3 on which $i \gg N$, $\Phi_i \gg 1$. The contribution of part 1 to $\mathbf{A}$ reduces approximatively to the term proportional to $\mathbf{a}_1$ (the contribution of the ingoing particle) since if one puts $\Phi_i = 0$, all $\mathbf{a}_i$'s on the inner segments of trajectory cancel. The contribution of part 3 to $\mathbf{A}$ is negligible because of rapid oscillations of the phase factor there. Finally, the contribution of part 2 to $\mathbf{A}$ can be roughly estimated by the term proportional to $\mathbf{a}_N$ with $\Phi_N = 0$ (the contribution of the outgoing particle). Thus one can compute $\mathbf{A}$ in such a way that one ignores all the terms with $i > N$ and puts $\Phi_i = 0$ in all the terms with $i \leq N$. This is called the factorization limit. In this limit the quantity $\mathbf{A}$ is the same as by the scattering on an isolated centre, i.e. it is given by Eq's (7) and (8) with the final velocity $\mathbf{v}' = \mathbf{v}_N$. The limit consists in factorizing out all intermediate scatterings, so that the particle is considered as changing its velocity from $\mathbf{v}$ to $\mathbf{v}'$ instantenously.

It is easy to see that the distance that the particle "skips" in the factorization limit is just the formation length. In fact, the reasoning leading to the factorization limit refines the argument by which the formation length has been introduced. As we now see, not only the radiation coming from within the formation length is suppressed, but the same is true also about the radiation coming from outside of the formation length. Only the segments of trajectory at both ends of the formation length contribute to the radiation significantly. One may say that the radiation is formed gradually and the process is completed just when the formation length is crossed.

5 How a noncoherence arises from a coherence

Let us now look at the consequences of the factorization limit. If the particle is nonrelativistic, the denominators in $\mathbf{a}$ and $\mathbf{a}'$ can be replaced by 1, so that

$$\mathbf{A} \approx -\frac{1}{i\omega}\mathbf{n} \times \Delta\mathbf{v}$$

where $\Delta\mathbf{v} \equiv \mathbf{v}' - \mathbf{v}$. For small scattering angles, $\Delta\mathbf{v}$ is just the shift of the particle in the velocity space in a plane orthogonal to vector $\mathbf{v}$. This shift changes with time by the 2-dimensional "walk of a drunken sailor". Consequently,

$$\overline{(\mathbf{n} \times \Delta\mathbf{v})^2} = N < (\mathbf{n} \times \Delta\mathbf{v}_{ind})^2 >= N(1 - \frac{1}{2}\sin^2\chi) < \alpha_{ind}^2 >,$$

where χ is the angle between the vectors $\mathbf{n}$ and $\mathbf{v}$. ¿From the first equality it follows

$$\overline{|\mathbf{A}|^2} \approx N < |\mathbf{A}_{ind}|^2 >, \tag{11}$$

which is just the condition of noncoherence.

If the particle is ultrarelativistic, situation is somewhat more complicated since the denominators in $\mathbf{a}$ and $\mathbf{a}'$ can be close to zero. Suppose, as in the argument by Landau and Pomeranchuk, that vector $\mathbf{n}$ has the direction of initial velocity. Then $\mathbf{A}$ reduces to

$$\mathbf{A} = -\frac{\mathbf{n} \times \Delta\mathbf{v}}{i\omega(1 - \mathbf{n}.\mathbf{v}')}.$$

Using once more the "law of walking drunken sailor" one can cast the mean phase from the argument by Landau and Pomeranchuk into the form

$$\overline{\Phi} = \omega(\gamma^{-2} + \frac{1}{2}\overline{\alpha^2})t.$$

¿From this it can be seen that the critical frequency is just the frequency at which the particle passes the cone of emission. Note that this justifies the procedure by which the critical frequency has been established, since it mea ns that Eq. (3) for the formation length has been used legally. At frequencies higher than critical the particle remains in the cone of emission all the time, so that $1 - \mathbf{n}.\mathbf{v}' \doteq \gamma^{-2}$ and

$$\overline{\left(\frac{\mathbf{n} \times \Delta\mathbf{v}}{1 - \mathbf{n}.\mathbf{v}'}\right)^2} = N\gamma^4 < (\mathbf{n} \times \Delta\mathbf{v}_{ind})^2 >= N\gamma^4 < \alpha_{ind}^2 > .$$

In this case Eq. (11) holds again and radiation is effectively noncoherent. On the other hand, at frequencies lower than critical the particle crosses the cone of emission before reaching the formation length. In this case $1 - \mathbf{n}.\mathbf{v}' \doteq \alpha^2 \gg \gamma^{-2}$ and instead of Eq. (11) we have

$$\overline{|\mathbf{A}|^2} \ll N < |\mathbf{A}_{ind}|^2 >,$$

so that radiation is coherently damped.

It might seem that since all contributions to the radiation coming from inner segments of the trajectory of the particle are suppresed, the radiation must be suppressed, too. However, this is true neither for a nonrelativistic particle, nor for an ultrarelativistic particle radiating at frequencies above critical. The inspection of the factorization limit shows why it is so. The reason is that destructive inteference that takes place on inner segments of the trajectory is balanced by constructive interference between outer segments. The idea can be demonstrated on the expressions $(1 - 2 + 2 - 3 + ... + 100)^2$ and $(1 - 2)^2 + (2 - 3)^2 + ... + (99 - 100)^2$. The first expression (coherently squared sum of the sequence $1 - 2$, $2 - 3$, ... , $99 - 100$) is greater than the second one (noncoherently squared sum of the same sequence) although all its inner terms cancel.

6 The $\omega^{1/2}$ law

With the help of previous formulae it is easy to determine how the energy radiated into the solid angle 4π depends on frequency. Estimations similar to those that will follow can be found in [7].

The energy per unit interval of frequencies and per unit length passed by the particle can be written as

$$\frac{d\epsilon}{d\omega ds} \approx \frac{q^2\omega^2}{Nl}\overline{|\mathbf{A}|^2}\Delta\Omega, \tag{12}$$

where $\mathbf{A}$ is taken at $\mathbf{n} \uparrow\uparrow \mathbf{v}$ and $\Delta\Omega$ is the solid angle inside of which $\mathbf{A}(\mathbf{n}) \approx \mathbf{A}$. If the vector $\mathbf{n}$ is declined from the vector $\mathbf{v}$ by an angle χ, it holds

$$1 - \mathbf{n}.\mathbf{v}' \doteq \frac{1}{2}[\gamma^{-2} + (\vec{\chi} - \vec{\alpha})^2].$$

Since the expression on the left hand side enters $\mathbf{A}(\mathbf{n})$ in a negative power, χ is effectively truncated at $\underline{\max}(\gamma^{-1}, \alpha)$. As a result, $\Delta\Omega$ is of order γ^{-2} at frequencies higher than critical, and of order $\overline{\alpha^2}$ at frequencies lower than critical. In the former case

$$\overline{|\mathbf{A}|^2} \propto \frac{N}{\omega^2},$$

so that the dependence of the right hand side of Eq. (12) on frequency, both explicit and implicit through N, cancels out and the whole expression is frequency independent. This is just what the Bethe-Heitler formula says. In the latter case

$$\overline{|\mathbf{A}|^2} \propto \frac{1}{\omega^2}\overline{\frac{1}{\alpha^2}}.$$

Combining this with $\Delta\Omega \approx \overline{\alpha^2}$, we see that the energy of radiation depends on frequency neither explicitly nor implicitly through α; however, it depends on frequency implicitly through N. Since the energy of radiation is inversely proportional to N, and since $N = l_f/l$, using Eq. (3) one would obtain that the energy of radiation is proportional to ω. However, the particle gets off of the cone of emission before it passes the formation length, so that Eq. (3) does not hold. To fix the result, the condition $\overline{\Phi}(l_f) = 1$ can be used. Neclecting the first term in round brackets in Eq. (5), one finds that formation length is proportional to $\omega^{-1/2}$, so that energy is proportional to $\omega^{1/2}$. This is the modification of the Bethe-Heitler formula obtained by Landau and Pomeranchuk in the second paper of ref. [1].

7 Conclusion

The noncoherence of radiation at frequencies greater than critical is not, in fact, a generic property of the radiation as suggests the original argument by Landau and Pomeranchuk. It rather results from the specific form of "the law of walking drunken sailor" (the law that states that the rms of the distance passed by random walk is proportional to the square root of the number of steps). One can imagine that "the law of walking drunken sailor" is replaced by "the law of walking sober sailor" acording to which the particle declines from the original direction of motion to the same side at every scattering. Then the radiation, instead of being noncoherent above the critical frequency, will be coherently enhanced.

The description of radiation of a particle in a medium on the basis of Eq. (1) can be, at least qualitatively, applied also on bremsstrahlung in systems of particles and antiparticles, such as the quark gluon plasma. In such systems LPM effect has to occure not only in bremsstrahlung, but also in one-photon creations and anihilations of pairs. These processes were considered by Migdal in the second paper of ref. [2]. Both creations and anihilations start as noncoherent at the threshold, i.e. when the antiparticle in the initial or final state is at rest, and become coherent at sufficiently high energies of photon. The effect is, in a sense, complementary to that considered here. While a photon produced by bremsstrahlung does not "feel" subsequent collisions of the emitting particle inside a region where it is formed, a photon produced or absorbed in a creation or an anihilation "feels" subsequent collisions far outside such a region.

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