SOME REMARKS ABOUT THE GLAUBER-SUDARSHAN QUASIPROBABILITY[1]

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It is shown how two representations of the Glauber-Sudarshan quasiprobability in the Fock-state basis which are different in the form are related to each other and to the Sudarshan representation by the relations between the two-dimensional and the central-symmetric one-dimensional delta function and their derivatives. The regularized representation of the Glauber-Sudarshan quasiprobability by Peřina and Mišta is reconsidered in new light and is represented by introduction of the Laguerre $2D$-functions. A new representation of the Glauber-Sudarshan quasiprobability by Hermite polynomials with a differentiation operator in the argument acting onto a one-dimensional delta function is found. The connections to representations of more generally ordered quasiprobabilities are established. It is shown that the convolution of the Glauber-Sudarshan quasiprobabilities for Fock states does not lead to a new Glauber-Sudarshan quasiprobability for a "physical" state corresponding to positively definite density operators. The Appendices present the derivation of identities including generalized functions which are relevant for many calculations in quantum optics.

1. Introduction

The Glauber-Sudarshan quasiprobability [1–3] is the most singular quasiprobability among the quasiprobabilities used in quantum optics. In particular, the Glauber-Sudarshan quasiprobability of the Fock states is given by derivatives of the two-dimensional delta function. An expansion of states in Fock states leads therefore to an expansion in delta functions and their derivatives which is a series which convergence has to be considered in the sense of weak convergence of generalized functions that means as continuous linear functionals over a space of "sufficiently well-behaved" functions. In this sense, the Glauber-Sudarshan quasiprobability exists for arbitrary normalizable states

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and discussions about the nonexistence of the Glauber-Sudarshan quasiprobability for certain classes of states in the early time after its introduction can be considered as widely clarified [4–6].

We consider in this paper two different representations of the Glauber-Sudarshan quasiprobability which distinguish themselves by the presence or absence of a Gaussian function in front of the derivatives of the delta functions. We clarify the connection between these two possible representations in Sections 2 and 3 and present this as a limiting case of the more general s–ordered quasiprobabilities [7,8] embedded into the complete Gaussian class of quasiprobabilities [9]. The representation given by Sudarshan [1] is a representation, where the two-dimensional delta function is substituted by a one-dimensional central-symmetric delta function combined with angle functions but there remains the Gaussian factor in front of the derivatives of this delta function. The "disentanglement" of this product of Gaussian factor with derivatives of the delta function leads to a new representation derived in Section 4. We introduce in Section 5, in analogy to the one-dimensional set of Hermite functions, a two-dimensional orthonormalized and complete set of functions which we call Laguerre $2D$-functions and discuss some of its properties. This set is appropriate for the Fock-state representation of quasiprobabilities and for the inversion of the Fock-state matrix elements from the quasiprobabilities. In Section 6, we reconsider the Peřina-Mišta representation [10–12] and [5,6] of the Glauber-Sudarshan quasiprobability which is a regularized representation in comparison to the other considered representations. In Section 7, we consider the convolution of two Glauber-Sudarshan quasiprobabilities for two states as a new Glauber-Sudarshan quasiprobability of a new state and show that the density operator of this new state is, in general, not positively definite. Therefore, this convolution [2,3] cannot be generalized to a principle for arbitrary states. We show explicitly the violation of the positive definiteness of the resulting density operator for the combination of Fock states in the mentioned sense. In the Appendices, we derive some formulas to generalized functions which are important for many calculations in quantum optics and, in particular, in the present paper but which can be rarely found in this ready form in monographs about generalized functions.

2. Quasiprobabilities in Fock-state representation

We begin with the Fock-state representation of the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ expressed by an expansion in the two-dimensional delta function $\delta(\alpha, \alpha^*)$ and its derivatives which we prefer in comparison to the representation by the central-symmetric one-dimensional delta function and its derivatives given by Sudarshan [1]. Furthermore, we consider the relations to other quasiprobabilities.

The Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ is associated with normal ordering of the transition operator from the density operator ϱ to the quasiprobability and can be defined in the following way [1–8,13]

$$P(\alpha, \alpha^*) \quad = \quad \left\langle \varrho \mathcal{N} \left\{ \exp\left(-a \frac{\partial}{\partial \alpha} - a^\dagger \frac{\partial}{\partial \alpha^*} \right) \right\} \right\rangle \delta(\alpha, \alpha^*)$$

$$
\begin{aligned}
&= \left\langle \varrho \exp\left(-a^{\dagger}\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)\right\rangle \delta(\alpha,\alpha^*) \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}}{k!l!}\left\langle a^k\varrho\, a^{\dagger\, l}\right\rangle \frac{\partial^{k+l}}{\partial\alpha^k\partial\alpha^{*\, l}}\delta(\alpha,\alpha^*), \qquad (2.1)
\end{aligned}
$$

where $\mathcal{N}\{\ldots\}$ is the symbol for normal ordering of the content in braces and $\langle A\rangle \equiv \mathrm{Trace}(A)$ the very convenient notation for the trace of an operator A. The reconstruction of the density operator ϱ from the Glauber-Sudarshan quasiprobability $P(\alpha,\alpha^*)$ is obtained by

$$
\varrho = \int \frac{i}{2} d\alpha\wedge d\alpha^* P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|, \qquad (2.2)
$$

which is often taken as the starting point for the definition of $P(\alpha,\alpha^*)$. Herein as usual, $|\alpha\rangle$ denotes the normalized coherent states. The equivalence is evident from the relation

$$
\langle\beta|\exp\left(-a^{\dagger}\frac{\partial}{\partial\alpha^*}\right)\exp\left(-a\frac{\partial}{\partial\alpha}\right)|\beta\rangle\,\delta(\alpha,\alpha^*) = \delta(\alpha-\beta,\alpha^*-\beta^*). \qquad (2.3)
$$

The relation (2.1) is very convenient to obtain the Fock-state representation of the Glauber-Sudarshan quasiprobability by means of the expansion

$$
\varrho = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|m\rangle\langle m|\varrho|n\rangle\langle n|. \qquad (2.4)
$$

By inserting this expansion into (2.1) and by calculating the arising matrix elements of the transition operator, one finds [13] (Eq.(8.19))

$$
\begin{aligned}
P(\alpha,\alpha^*) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\frac{(-1)^{m+n}}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\frac{\partial^{m+n-2j}}{\partial\alpha^{m-j}\partial\alpha^{*\,n-j}}\delta(\alpha,\alpha^*) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\sqrt{\frac{n!}{m!}}\left(-\frac{\partial}{\partial\alpha}\right)^{m-n} L_n^{m-n}\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta(\alpha,\alpha^*), \qquad (2.5)
\end{aligned}
$$

where $L_n^{\nu}(u)$ denotes the Laguerre polynomials in the convention of Bateman and Erdelýi [14] (see remark in Appendix A of [15]). For Fock states $\varrho = |n\rangle\langle n|$, one obtains ($L_n(u)\equiv L_n^0(u)$)

$$
\varrho = |n\rangle\langle n|, \quad \Leftrightarrow \quad P(\alpha,\alpha^*) = L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta(\alpha,\alpha^*). \qquad (2.6)
$$

The representation of the quasiprobability $P(\alpha,\alpha^*)$ by generalized functions given by Sudarshan [1] is different from the representation given in (2.5). We come back to this point in the next two Sections. Now, we will show how (2.5) can be generalized to the complete Gaussian class of quasiprobabilities considered in [9]. For this purpose we mention that the quasiprobabilities $F_{(r_1,r_2,r_3)}(\alpha,\alpha^*)$ of this class can be obtained from the Wigner quasiprobability $W(\alpha,\alpha^*)$ by the convolution (notation $*$ for the operation of forming the convolution; furthermore, note $W(\alpha,\alpha^*) \equiv F_{(0,0,0)}(\alpha,\alpha^*)$)

$$F_{(r_1,r_2,r_3)}(\alpha,\alpha^*) = g_{(r_1,r_2,r_3)}(\alpha,\alpha^*) * W(\alpha,\alpha^*), \tag{2.7}$$

where $g_{(r_1,r_2,r_3)}(\alpha,\alpha^*)$ denotes the following normalized centered two-dimensional Gaussian function

$$\begin{aligned} g_{(r_1,r_2,r_3)}(\alpha,\alpha^*) &= \frac{2}{\pi\sqrt{\boldsymbol{r}^2}} \exp\left\{-\frac{(r_1+ir_2)\alpha^2 - (r_1-ir_2)\alpha^{*\,2} + 2r_3\alpha\alpha^*}{\boldsymbol{r}^2}\right\}, \\ \boldsymbol{r}^2 &= r_1^2 + r_2^2 + r_3^2. \end{aligned} \tag{2.8}$$

The Glauber-Sudarshan quasiprobability $P(\alpha,\alpha^*)$ is related to the Wigner quasiprobability $W(\alpha,\alpha^*)$ by (note $P(\alpha,\alpha^*) \equiv F_{(0,0,-1)}(\alpha,\alpha^*)$)

$$P(\alpha,\alpha^*) = g_{(0,0,-1)}(\alpha,\alpha^*) * W(\alpha,\alpha^*), \qquad W(\alpha,\alpha^*) = g_{(0,0,1)}(\alpha,\alpha^*) * P(\alpha,\alpha^*). \tag{2.9}$$

Since the introduced vector parameter $\boldsymbol{r} \equiv (r_1,r_2,r_3)$ is additive with regard to convolutions [9] meaning $g_{\boldsymbol{r}}(\alpha,\alpha^*) * g_{\boldsymbol{s}}(\alpha,\alpha^*) = g_{\boldsymbol{r}+\boldsymbol{s}}(\alpha,\alpha^*)$, one obtains from (2.7) and (2.9)

$$F_{(r_1,r_2,r_3)}(\alpha,\alpha^*) = g_{(r_1,r_2,1+r_3)}(\alpha,\alpha^*) * P(\alpha,\alpha^*), \tag{2.10}$$

This means that the Fock-state representation of the quasiprobability $F_{(r_1,r_2,r_3)}(\alpha,\alpha^*)$ can be obtained from (2.5) in the following form

$$\begin{aligned} &F_{(r_1,r_2,r_3)}(\alpha,\alpha^*) \\ =\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle \\ &\times \frac{(-1)^{m+n}}{\sqrt{m!n!}} \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \frac{\partial^{m+n-2j}}{\partial\alpha^{m-j}\partial\alpha^{*\,n-j}} g_{(r_1,r_2,1+r_3)}(\alpha,\alpha^*) \\ =\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle \sqrt{\frac{n!}{m!}} \left(-\frac{\partial}{\partial\alpha}\right)^{m-n} L_n^{m-n}\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) g_{(r_1,r_2,1+r_3)}(\alpha,\alpha^*), \end{aligned} \tag{2.11}$$

The only difference on the right-hand side to (2.5) is that the two-dimensional delta function $\delta(\alpha,\alpha^*)$ is substituted by the normalized Gaussian function $g_{(r_1,r_2,1+r_3)}(\alpha,\alpha^*)$. This means for Fock states

$$\varrho = |n\rangle\langle n|, \quad \Leftrightarrow \quad F_{(r_1,r_2,r_3)}(\alpha,\alpha^*) = L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) g_{(r_1,r_2,1+r_3)}(\alpha,\alpha^*). \tag{2.12}$$

It is clear that it is not easy to calculate the necessary derivatives of $g_{(r_1,r_2,1+r_3)}(\alpha,\alpha^*)$ in the general case and we will explicitly consider this only for the $(0,0,r)$–ordered class of quasiprobabilities (usually called s–ordered class with $r=-s$; we have some reason to change the sign [9] but this is not a deep problem).

For the $(0,0,r)$–ordered class of quasiprobabilities, one easily finds the specialization $g_{(0,0,1+r)}(\alpha,\alpha^*)$ from (2.8) and by inserting this into (2.11), one obtains

$$\begin{aligned}
&F_{(0,0,r)}(\alpha,\alpha^*)\\
=\;&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\\
&\times\frac{(-1)^{m+n}}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\frac{\partial^{m+n-2j}}{\partial\alpha^{m-j}\partial\alpha^{*\,n-j}}\frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\\
=\;&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\sqrt{\frac{n!}{m!}}\left(-\frac{\partial}{\partial\alpha}\right)^{m-n}L_n^{m-n}\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right),
\end{aligned}\tag{2.13}$$

in particular, for Fock states

$$\varrho=|n\rangle\langle n|,\quad\Leftrightarrow\quad F_{(0,0,r)}(\alpha,\alpha^*)=L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right).\tag{2.14}$$

By inserting the result of the differentiations of the Gaussian function as derived in Appendix A with $\sigma=(1+r)/2$, one obtains (derived in [7] in another way)

$$\begin{aligned}
&F_{(0,0,r)}(\alpha,\alpha^*)\\
=\;&\frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\\
&\times\frac{1}{\sqrt{m!n!}}\left(\frac{2}{1+r}\right)^{m+n}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(-\frac{1-r^2}{4}\right)^{j}\alpha^{*\,m-j}\alpha^{n-j}\\
=\;&\frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\\
&\times\sqrt{\frac{n!}{m!}}\left(-\frac{1-r}{1+r}\right)^{n}\left(\frac{2\alpha^*}{1+r}\right)^{m-n}L_n^{m-n}\left(\frac{4\alpha\alpha^*}{1-r^2}\right),
\end{aligned}\tag{2.15}$$

in particular, for Fock states

$$\varrho=|n\rangle\langle n|,\quad\Leftrightarrow\quad F_{(0,0,r)}(\alpha,\alpha^*)=\frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\left(-\frac{1-r}{1+r}\right)^{n}L_n\left(\frac{4\alpha\alpha^*}{1-r^2}\right).\tag{2.16}$$

The coherent-state quasiprobability $Q(\alpha,\alpha^*)$ is obtained as the special case $r=1$. In this case, we can use the limiting procedure $n!\lim_{u\to 0}L_n^{\nu}(u)/(-u)^n=1$ and find from (2.15)

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\exp\left(-\alpha\alpha^*\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\frac{\alpha^{*\,m}\alpha^{n}}{\sqrt{m!n!}},\tag{2.17}$$

in particular, for Fock states

$$\varrho = |n\rangle\langle n|, \quad \Leftrightarrow \quad Q(\alpha,\alpha^*) = \frac{1}{\pi}\exp\left(-\alpha\alpha^*\right)\frac{(\alpha\alpha^*)^n}{n!}. \tag{2.18}$$

The Wigner quasiprobability $W(\alpha,\alpha^*)$ is obtained as the special case $r = 0$ and one finds from (2.15)

$$\begin{aligned} W(\alpha,\alpha^*) &= \frac{2}{\pi}\exp\left(-2\alpha\alpha^*\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle \\ &\quad\times\frac{2^{m+n}}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\left(-\frac{1}{4}\right)^j\alpha^{*\,m-j}\alpha^{n-j} \\ &= \frac{2}{\pi}\exp\left(-2\alpha\alpha^*\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle(-1)^n\sqrt{\frac{n!}{m!}}\left(2\alpha^*\right)^{m-n}L_n^{m-n}\left(4\alpha\alpha^*\right), \end{aligned} \tag{2.19}$$

in particular, for Fock states [2,3]

$$\varrho = |n\rangle\langle n|, \quad \Leftrightarrow \quad W(\alpha,\alpha^*) = \frac{2}{\pi}\exp\left(-2\alpha\alpha^*\right)(-1)^n L_n(4\alpha\alpha^*). \tag{2.20}$$

In case of the Glauber-Sudarshan quasiprobability corresponding to $r = -1$, the argument of the Laguerre polynomials goes to infinity and there are no factors in front of the Laguerre polynomials which compensate these divergencies. Therefore, the explicit representation (2.15) is not applicable in this case and one has to go back to the representation (2.5) involving derivatives of the delta function which formed the starting point of our considerations.

We mention here that for displaced Fock states $|\beta,n\rangle$ defined by $|\beta,n\rangle = D(\beta,\beta^*)|n\rangle$ with $D(\beta,\beta^*)$ as the unitary displacement operator (see, e.g. [16,13]) holds the following basic relation to density operators of coherent states [13]

$$|\beta,m\rangle\langle\beta,n| = \sqrt{\frac{n!}{m!}}\frac{\partial^{m-n}}{\partial\beta^{m-n}}L_n^{m-n}\left(-\frac{\partial^2}{\partial\beta\partial\beta^*}\right)|\beta\rangle\langle\beta| \tag{2.21}$$

It can be obtained from the coherent-state quasiprobability for the Fock-state operators $|m\rangle\langle n|$ according to (2.13) for $r = 1$ in the following way

$$\begin{aligned} \frac{1}{\pi}\langle\alpha|\beta,m\rangle\langle\beta,n|\alpha\rangle &= \frac{1}{\pi}\langle\alpha-\beta|m\rangle\langle n|\alpha-\beta\rangle \\ &= \sqrt{\frac{n!}{m!}}\left(-\frac{\partial}{\partial\alpha}\right)^{m-n}L_n^{m-n}\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\frac{1}{\pi}\langle\alpha-\beta|0\rangle\langle 0|\alpha-\beta\rangle \\ &= \sqrt{\frac{n!}{m!}}\left(\frac{\partial}{\partial\beta}\right)^{m-n}L_n^{m-n}\left(-\frac{\partial^2}{\partial\beta\partial\beta^*}\right)\frac{1}{\pi}\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle, \end{aligned} \tag{2.22}$$

which proves (2.21) since both sides are quasiprobabilities from which the operator $|\beta, m\rangle\langle\beta, n|$ is uniquely determined. Due to the displacement structure of quasiprobabilities, an analogous proof could be made with arbitrary other quasiprobabilities. The relation (2.21) was proved in [13] by recursion relations for the displaced Fock-state operators.

3. Alternative form of the Fock-state representation of the Glauber-Sudarshan quasiprobability

We now consider an alternative form of the Glauber-Sudarshan quasiprobability in Fock-state representation which is near to the representation given by Sudarshan [1].

Starting point of the considerations of this Section is relation (2.2) from which we form the Fock-state matrix elements

$$\langle m|\varrho|n\rangle = \int \frac{i}{2} d\alpha \wedge d\alpha^* P(\alpha, \alpha^*) \exp\left(-\alpha\alpha^*\right) \frac{\alpha^m \alpha^{*\,n}}{\sqrt{m!n!}}. \tag{3.1}$$

On the right-hand side, we have the moments of the function $P(\alpha, \alpha^*) \exp(-\alpha\alpha^*)$. From the reconstruction formula of a function from its moments (see Appendix B), we find the following expansion of $P(\alpha, \alpha^*)$

$$P(\alpha, \alpha^*) = \exp\left(\alpha\alpha^*\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \frac{(-1)^{m+n}}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial\alpha^m \partial\alpha^{*\,n}} \delta(\alpha, \alpha^*). \tag{3.2}$$

If we compare this with (2.5), we find the following identity

$$\begin{aligned} \exp(\alpha\alpha^*) \frac{\partial^{m+n}}{\partial\alpha^m \partial\alpha^{*\,n}} \delta(\alpha, \alpha^*) &= \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \frac{\partial^{m+n-2j}}{\partial\alpha^{m-j} \partial\alpha^{*\,n-j}} \delta(\alpha, \alpha^*) \\ &= n! \frac{\partial^{m-n}}{\partial\alpha^{m-n}} L_n^{m-n}\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) \delta(\alpha, \alpha^*). \end{aligned} \tag{3.3}$$

The difference between the expansions (2.5) and (3.2) of the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ is that in (3.2) we have the function $\exp(\alpha\alpha^*)$ in front of the derivatives of the delta function. As a rule, it is a complicated operation to have a function of a variable in front of derivatives of a delta function of the same variable as can be seen from the example (3.3) and one can give a general resolution in form of linear combinations of lower derivatives of the delta function with number coefficients in front of these derivatives ("disentanglement" of such products; see Appendix C).

The form (3.2) of the Glauber-Sudarshan quasiprobability is favourable in calculations where the factor $\exp\left(\alpha\alpha^*\right)$ on the right-hand side is absorbed and there remain the "pure" derivatives of the delta functions. In the other cases, one has mostly to prefer the form (2.5) for the Glauber-Sudarshan quasiprobability in Fock-state representation.

4. Sudarshan representation of the Glauber-Sudarshan quasiprobability

The Sudarshan representation of the Glauber-Sudarshan quasiprobability $P(\alpha, \alpha^*)$ in the Fock-state basis uses the one-dimensional delta function and its derivatives combined with simple angle functions instead of the more basic two-dimensional delta function $\delta(\alpha, \alpha^*)$ for the phase space of one mode. We represent the transition from the two-dimensional delta function and its derivative as used in the preceding Sections to a one-dimensional central-symmetric delta function in Appendix D. One has to observe one moment of this representation. The integration over the two-dimensional phase space in polar coordinates $(|\alpha|, \varphi)$ includes the integration over $|\alpha|$ from 0 to $+\infty$. This integration is not uniquely determined if one has the one-dimensional central-symmetric delta function $\delta(|\alpha|)$ in the integrand because the singularity is at one border of the integrations. It seems to us that this can be taken into account in the best way by a limiting procedure which one has to make at the end of calculating functionals over the generalized functions and we represent this in such a way. In all other moments we get results in accordance with [1].

If we insert the identity (D.6) of Appendix D into the formula (3.2), we obtain the following representation of the Glauber-Sudarshan quasprobability

$$P(\alpha, \alpha^*) = \exp\left(|\alpha|^2\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \frac{(-1)^{m+n}\sqrt{m!n!}}{(m+n)!} \frac{\mathrm{e}^{i(n-m)\varphi}}{2\pi|\alpha|} \lim_{\varepsilon \to 0} \frac{\partial^{m+n}}{\partial|\alpha|^{m+n}} \delta(|\alpha| - \varepsilon). \tag{4.1}$$

Apart from the limiting procedure, this is in agreement with the representation given by Sudarshan [1] (comp. [2,3]), who probably solved some problems of the transition from the two-dimensional representation of the delta function to the one-dimensional representation which we present in Appendix D and which can be rarely found in monographs about generalized functions (some results can be "guessed" by considering simple "test" functions as, for example, power functions). An alternative representation can be derived by inserting (D.6) of Appendix D into formula (2.5) that leads to

$$\begin{aligned} P(\alpha, \alpha^*) \quad &= \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \frac{(-1)^{m+n}\sqrt{m!n!}}{(m+n)!} \frac{\mathrm{e}^{i(n-m)\varphi}}{2\pi|\alpha|} \\ &\times \lim_{\varepsilon \to 0} \sum_{j=0}^{[\frac{m+n}{2}]} \frac{(m+n)!}{j!(m+n-2j)!} \frac{\partial^{m+n-2j}}{\partial|\alpha|^{m+n-2j}} \delta(|\alpha| - \varepsilon), \end{aligned} \tag{4.2}$$

and can be represented by Hermite polynomials $H_k(x)$ as follows

$$P(\alpha, \alpha^*) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \frac{i^{m+n}\sqrt{m!n!}}{(m+n)!} \frac{\mathrm{e}^{i(n-m)\varphi}}{2\pi|\alpha|} \lim_{\varepsilon \to 0} H_{m+n}\left(\frac{i}{2} \frac{\partial}{\partial|\alpha|}\right) \delta(|\alpha| - \varepsilon). \tag{4.3}$$

The direct connection between the two representations (4.1) and (4.3) can be established by using the identity (C.6) in Appendix C with $c = i$.

One can consider (4.3) as the disentangled version of the Sudarshan representation of the quasiprobability $P(\alpha, \alpha^*)$. It seems to be favourable to use it in calculations when there is no Gaussian factor $\exp\left(-|\alpha|^2\right)$ which compensates the opposite Gaussian factor in the Sudarshan representation (4.1) but one has to decide this in every concrete case. Both representations, the Sudarshan representation and the representation (4.2) or (4.3) are favourable to use if one has problems which are appropriate to solve in polar coordinates. In the other cases, however, we prefer to use the representations by two-dimensional delta functions considered in (2.5) and (3.2).

5. Introduction of Laguerre 2D-functions

We introduce the following set of functions of two variables in complex representation

$$\begin{aligned} l_{m,n}(z,z^*) &= \frac{1}{\sqrt{\pi}}\exp\left(-\frac{zz^*}{2}\right)(-1)^n\sqrt{\frac{n!}{m!}}z^{m-n}L_n^{m-n}(zz^*) \\ &= \frac{1}{\sqrt{\pi}}\exp\left(-\frac{zz^*}{2}\right)(-1)^m\sqrt{\frac{m!}{n!}}z^{*\,n-m}L_m^{n-m}(zz^*), \end{aligned} \tag{5.1}$$

and call it Laguerre 2D-functions (two-dimensional Laguerre functions contrary to usual one-dimensional Laguerre functions) in analogy to the orthonormalized Hermite functions $h_n(x)$ playing a similar role as a basis for functions of one variable. The more explicit representation of the Laguerre 2D-functions is

$$l_{m,n}(z,z^*) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{zz^*}{2}\right)\frac{1}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}(-1)^j z^{m-j}z^{*\,n-j}. \tag{5.2}$$

One immediately finds the symmetry properties

$$l_{m,n}(z,z^*) = \left(l_{n,m}(z,z^*)\right)^* = l_{n,m}(z^*,z), \qquad l_{m,n}(-z,-z^*) = (-1)^{m+n}l_{m,n}(z,z^*). \tag{5.3}$$

The set of Laguerre 2D-functions is orthonormalized in the following way

$$\int \frac{i}{2}dz\wedge dz^*\left(l_{k,l}(z,z^*)\right)^* l_{m,n}(z,z^*) = \delta_{k,m}\delta_{l,n}, \tag{5.4}$$

and obeys the following completeness relation

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} l_{m,n}(z,z^*)\left(l_{m,n}(w,w^*)\right)^* = \delta(z-w,z^*-w^*). \tag{5.5}$$

The orthonormality relations can be proved in modified polar coordinates $(u=|z|^2,\varphi)$ (action-angle coordinates), where the first integration over φ leads to a known special integral over products of two Laguerre polynomials combined with exponential and power functions (proof, e.g., in [17]). In a similar way, in polar coordinates for both complex variables z and w, the completeness relation (5.5) can be proved. One can first separate a sum which contains only the moduli $|z|$ and $|w|$ and which can be evaluated by a limiting procedure from a known sum (Eq.(20) chap.10.12 in [14], known as Hille-Hardy or Myller-Lebedeff formula) and then the remaining sum with the angles as parameters can be easily evaluated providing the delta function of the difference of the angles. If the function system $l_{m,n}(z,z^*)$ is complete that is intuitively clear, then the form (5.5) of the completeness relation follows automatically from the orthonormality relations (5.4).

By direct calculation, one proves that the Fourier transforms of the Laguerre $2D$-functions are again Laguerre $2D$-functions according to

$$\tilde{l}_{m,n}(w,w^*) \equiv \int \frac{i}{2} dz \wedge dz^* l_{m,n}(z,z^*) \exp\left(-\frac{i}{2}(w^*z + wz^*)\right) = 2\pi(-i)^{m+n} l_{m,n}(w,w^*), \tag{5.6}$$

analogously to the Hermite functions. The Radon transform of the Laguerre $2D$-functions is essentially the product of Hermite functions combined with angle functions

$$\begin{aligned} \breve{l}_{m,n}(w,w^*;c) &\equiv \int \frac{i}{2} dz \wedge dz^* l_{m,n}(z,z^*)\, \delta\left(c - \frac{1}{2}(w^*z + wz^*)\right) \\ &= \sqrt{\frac{2\pi}{ww^*}} \left(\sqrt{\frac{w}{w^*}}\right)^{m-n} h_m\left(\frac{c}{\sqrt{2ww^*}}\right) h_n\left(\frac{c}{\sqrt{2ww^*}}\right), \\ h_n(x) &= \frac{1}{\pi^{\frac{1}{4}}} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2^n n!}} H_n(x). \end{aligned} \tag{5.7}$$

A further relation is given in Appendix A. One of the differential equations to which the Laguerre $2D$-functions are solutions is the eigenvalue equation of a two-dimensional degenerate harmonic oscillator.

By using the Laguerre $2D$-functions, the quasiprobabilities $F_{(0,0,r)}(\alpha,\alpha^*)$ which are explicitly given in (2.15) can be represented by

$$\begin{aligned} F_{(0,0,r)}(\alpha,\alpha^*) &= \frac{2}{\sqrt{\pi}(1+r)} \exp\left(\frac{2r\,\alpha\alpha^*}{1-r^2}\right) \\ &\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \left(\sqrt{\frac{1-r}{1+r}}\right)^{m+n} l_{n,m}\left(\frac{2\alpha}{\sqrt{1-r^2}}, \frac{2\alpha^*}{\sqrt{1-r^2}}\right), \end{aligned} \tag{5.8}$$

in particular, the Wigner quasiprobability as the special case $r = 0$

$$W(\alpha,\alpha^*) = \frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle m|\varrho|n\rangle\, l_{n,m}(2\alpha, 2\alpha^*). \tag{5.9}$$

The normalization of the quasiprobabilities in this form can be proved from the relation

$$\int \frac{i}{2} dz \wedge dz^* \exp\left(\frac{r}{2} zz^*\right) l_{m,n}(z,z^*) = \frac{2\sqrt{\pi}}{1-r}\left(\frac{1+r}{1-r}\right)^n \delta_{m,n}. \tag{5.10}$$

The inversion of (5.8) yields for the Fock-state matrix elements

$$\begin{aligned}\langle m|\varrho|n\rangle &= \frac{2\sqrt{\pi}}{1-r}\left(\sqrt{\frac{1+r}{1-r}}\right)^{m+n}\int\frac{i}{2}d\alpha\wedge d\alpha^*\,F_{(0,0,r)}(\alpha,\alpha^*)\\ &\quad\times\exp\left(-\frac{2r\,\alpha\alpha^*}{1-r^2}\right)l_{m,n}\left(\frac{2\alpha}{\sqrt{1-r^2}},\frac{2\alpha^*}{\sqrt{1-r^2}}\right),\end{aligned}\tag{5.11}$$

in particular, the inversion from the Wigner quasiprobability

$$\langle m|\varrho|n\rangle = 2\sqrt{\pi}\int\frac{i}{2}d\alpha\wedge d\alpha^*\,W(\alpha,\alpha^*)\,l_{m,n}\,(2\alpha,2\alpha^*)\,.\tag{5.12}$$

By means of the well-known generating function of the Laguerre polynomials, one proves

$$\sum_{n=0}^{\infty}\left(\frac{1+r}{1-r}\right)^n l_{n,n}\,(z,z^*) = \frac{1-r}{2\sqrt{\pi}}\exp\left(\frac{r}{2}zz^*\right),\tag{5.13}$$

that guarantees the normalization $\langle\varrho\rangle = \sum_{n=0}^{\infty}\langle n|\varrho|n\rangle = 1$.

The above considerations show that the Laguerre $2D$-functions are very appropriate for the representation of the quasiprobabilities in the Fock-state basis and for their inversion but we think that they are useful also for many other purposes.

6. Peřina-Mišta representation of the Glauber-Sudarshan quasiprobability

Peřina and Mišta [10,11] (see also [5,6] and [12]) introduced a "regularized" representation of the Glauber-Sudarshan quasiprobability $P(\alpha,\alpha^*)$ by Laguerre polynomials to which is rarely paid attention up to now. We illuminate this representation here from a modified point of view and hope to contribute in this way to a better understanding and to its further application. For this purpose, we use the introduced Laguerre $2D$-functions.

By setting $r = -\sqrt{1-4\varepsilon}, \rightarrow \varepsilon = (1-r^2)/4$ in (5.8), one can represent the Glauber-Sudarshan quasiprobability by the following limiting procedure

$$\begin{aligned}P(\alpha,\alpha^*) &= \exp(\alpha\alpha^*)\lim_{\varepsilon\to 0}\left\{\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{\alpha\alpha^*}{2\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\langle m|\varrho|n\rangle}{(\sqrt{\varepsilon}\,)^{m+n}}\,l_{n,m}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right)\right\}\\ &= \exp(\alpha\alpha^*)\lim_{\varepsilon\to 0}\left\{\frac{1}{\pi\varepsilon}\exp\left(-\frac{\alpha\alpha^*}{\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\langle m|\varrho|n\rangle}{\sqrt{m!n!}\,\varepsilon^{m+n}}\right.\\ &\quad\left.\times\sum_{j=0}^{\{m,n\}}\frac{m!n!\,(-\varepsilon)^j}{j!(m-j)!(n-j)!}\alpha^{*\,m-j}\alpha^{n-j}\right\}.\end{aligned}\tag{6.1}$$

If we want to obtain a similar structure with the same Gaussian factors in front, however, without a limiting procedure, then we bring these factors to the left-hand side and by inserting (3.2), we find

$$\exp\left(\frac{\alpha\alpha^*}{2\varepsilon}\right)\exp(-\alpha\alpha^*)P(\alpha,\alpha^*)=\exp\left(\frac{\alpha\alpha^*}{2\varepsilon}\right)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle k|\varrho|l\rangle\frac{(-1)^{k+l}}{\sqrt{k!l!}}\frac{\partial^{k+l}}{\partial\alpha^k\partial\alpha^{*\,l}}\delta(\alpha,\alpha^*). \tag{6.2}$$

We now make an expansion of this expression in the set of functions $l_{n,m}\,(\alpha/\sqrt{\varepsilon},\alpha^*/\sqrt{\varepsilon})$. According to the completeness relation (5.5), we have to calculate the following integral of the right-hand side of (6.2)

$$\begin{aligned}
&\frac{1}{\varepsilon}\int\frac{i}{2}d\beta\wedge d\beta^* l_{m,n}\left(\frac{\beta}{\sqrt{\varepsilon}},\frac{\beta^*}{\sqrt{\varepsilon}}\right)\exp\left(\frac{\beta\beta^*}{2\varepsilon}\right)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle k|\varrho|l\rangle\frac{(-1)^{k+l}}{\sqrt{k!l!}}\frac{\partial^{k+l}}{\partial\beta^k\partial\beta^{*\,l}}\delta(\beta,\beta^*)\\
&=\frac{1}{\varepsilon}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle k|\varrho|l\rangle\frac{1}{\sqrt{k!l!}}\left\{\frac{\partial^{k+l}}{\partial\beta^k\partial\beta^{*\,l}}\left(\exp\left(\frac{\beta\beta^*}{2\varepsilon}\right)l_{m,n}\left(\frac{\beta}{\sqrt{\varepsilon}},\frac{\beta^*}{\sqrt{\varepsilon}}\right)\right)\right\}_{\beta=\beta^*=0}\\
&=\frac{1}{\sqrt{\pi}\,\varepsilon}\frac{1}{(\sqrt{\varepsilon}\,)^{m+n}}\sum_{j=0}^{\{m,n\}}\langle m-j|\varrho|n-j\rangle\frac{\sqrt{m!n!}\,(-\varepsilon)^j}{j!\sqrt{(m-j)!(n-j)!}}.
\end{aligned} \tag{6.3}$$

For the calculation of the derivatives at $\beta=\beta^*=0$ it was very favourable that the Gaussian factors in front of the Laguerre $2D$-functions and inside of the Laguerre $2D$-functions compensate each other and the expressions became easily calculable. This explains the choice of the Gaussian factors in (6.2) which is the best. Instead of (6.1), we now find the following regularized representation ($\varepsilon\geq 0$)

$$\begin{aligned}
&P(\alpha,\alpha^*)\\
=\;&\frac{1}{\sqrt{\pi}\,\varepsilon}\exp\left(\alpha\alpha^*-\frac{\alpha\alpha^*}{2\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varrho_{m,n}(\varepsilon)}{(\sqrt{\varepsilon}\,)^{m+n}}\,l_{n,m}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right)\\
=\;&\frac{1}{\pi\varepsilon}\exp\left(\alpha\alpha^*-\frac{\alpha\alpha^*}{\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varrho_{m,n}(\varepsilon)}{\sqrt{m!n!}\,\varepsilon^{m+n}}\sum_{j=0}^{\{m,n\}}\frac{m!n!\,(-\varepsilon)^j}{j!(m-j)!(n-j)!}\alpha^{*\,m-j}\alpha^{n-j},
\end{aligned} \tag{6.4}$$

with the following definition of the new matrix elements $\varrho_{m,n}(\varepsilon)$ together with their inversion (found by Peřina and coworkers [5,6])

$$\begin{aligned}
\varrho_{m,n}(\varepsilon)&\equiv\sum_{k=0}^{\{m,n\}}\frac{\sqrt{m!n!}\,(-\varepsilon)^k}{k!\sqrt{(m-k)!(n-k)!}}\varrho_{m-k,n-k}(0)=\sum_{k=0}^{\{m,n\}}\frac{(-\varepsilon)^k}{k!}\langle m|a^{\dagger\,k}\varrho\,a^k|n\rangle,\\
\varrho_{m,n}(0)&=\sum_{l=0}^{\{m,n\}}\frac{\sqrt{m!n!}\,\varepsilon^l}{l!\sqrt{(m-l)!(n-l)!}}\varrho_{m-l,n-l}(\varepsilon),\qquad \varrho_{m,n}(0)=\langle m|\varrho|n\rangle.
\end{aligned} \tag{6.5}$$

In (6.4) and (6.5), ε is a free parameter larger than zero. For $\varepsilon=0$, one obtains the singular representation (3.2). Note that in the representation which was obtained by Peřina and coworkers, there is used an older and now rarely used definition of the Laguerre polynomials [18] and that the representation is less symmetric by separation of the sum terms with $m > n$ and $m < n$ with introduction of $\mu \equiv \pm(m-n)$, where the terms with $m = n$ are joined with one of these sum terms. The representation given here was mainly obtained in this form by introduction of the Laguerre $2D$-functions $l_{m,n}(z, z^*)$ and due to their symmetries.

7. Convolution of Glauber-Sudarshan quasiprobabilities

A combination principle for two states to a new state of the following kind was put forward in [2,3]. The convolution of two quasiprobabilities $P_1(\alpha, \alpha^*)$ and $P_2(\alpha, \alpha^*)$ belonging to density operators ϱ_1 and ϱ_2 provides a new such quasiprobability $P(\alpha, \alpha^*)$ corresponding to a new state which could be considered as a kind of superposition of the two states that means

$$P'(\alpha, \alpha^*) = P_1(\alpha, \alpha^*) * P_2(\alpha, \alpha^*), \quad \Leftrightarrow \quad \varrho' = \mathrm{Conv}(\varrho_1, \varrho_2), \tag{7.1}$$

and its normalization is immediately to see. For example, the superposition of two coherent states $\varrho_1 = |\beta_1\rangle\langle\beta_1|$ and $\varrho_2 = |\beta_2\rangle\langle\beta_2|$ in this sense provides a new coherent state $\varrho' = |\beta_1 + \beta_2\rangle\langle\beta_1 + \beta_2|$ according to

$$\begin{aligned} P'(\alpha, \alpha^*) &= \delta(\alpha - \beta_1, \alpha^* - \beta_1^*) * \delta(\alpha - \beta_2, \alpha^* - \beta_2^*) \\ &= \delta(\alpha - \beta_1 - \beta_2, \alpha^* - \beta_1^* - \beta_2^*). \end{aligned} \tag{7.2}$$

We will show that, in general, this combination principle does not provide a positively definite Hermitean density operator ϱ'. This means that there cannot be an apparatus with two inputs for the states with the density operators ϱ_1 and ϱ_2 and, at least, one output for the state with the density operator ϱ' determined by a Glauber-Sudarshan quasiprobability $P'(\alpha, \alpha^*)$ according to (7.1). Although there are considerations to this failure in [4] (chap.8.4) which show that this combination principle cannot be claimed as a general principle, this is little known. The argumentation in [4] uses the Fourier transforms of the Glauber-Sudarshan quasiprobabilities which are to multiply if the quasiprobabilities themselves underly a convolution. We will give here an explicit example which shows that, in this way, we obtain resulting states corresponding to indefinite Hermitean "density" operators.

We consider two Fock states $\varrho_1 = |m\rangle\langle m|$ and $\varrho_2 = |n\rangle\langle n|$. The convolution of the corresponding Glauber-Sudarshan quasiprobabilities provides

$$\begin{aligned} P'(\alpha, \alpha^*) &= \left\{ L_m\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta(\alpha, \alpha^*)\right\} * \left\{ L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta(\alpha, \alpha^*)\right\} \\ &= L_m\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\int \frac{i}{2} d\beta \wedge d\beta^*\, \delta(\alpha - \beta, \alpha^* - \beta^*) L_n\left(-\frac{\partial^2}{\partial\beta\partial\beta^*}\right)\delta(\beta, \beta^*) \\ &= L_m\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) L_n\left(-\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\delta(\alpha, \alpha^*). \end{aligned} \tag{7.3}$$

The problem of determination of the Fock-state representation to this function considered as a Glauber-Sudarshan quasiprobability is the problem of the decomposition of the product of two Laguerre polynomials $L_m(u)L_n(u)$ into a sum over Laguerre polynomials

$$L_m(u)L_n(u) = \sum_{l=|m-n|}^{m+n} c_{m,n,l} L_l(u). \tag{7.4}$$

The restriction of the sum over Laguerre polynomials by an upper $l = m+n$ can be easily seen from the highest power of u in the product $L_m(u)L_n(u)$ and to a lower $l = |m-n|$ by the symmetry properties exposed below. Due to $L_n^\nu(0) = (n+\nu)!/(n!\nu!)$, one finds from (7.4) by setting $u = 0$

$$\sum_{l=|m-n|}^{m+n} c_{m,n,l} = \sum_{l=|m-n|}^{m+n} c_{m,n,l} L_l(0) = L_m(0)L_n(0) = 1. \tag{7.5}$$

This means that we have checked the normalization which is true, more generally, for arbitrary $P'(\alpha,\alpha^*)$ in (7.1). Due to the well-known orthonormality of the Laguerre polynomials with a weight function e^{-u}, one has

$$c_{m,n,l} = \int_0^{+\infty} du\, \mathrm{e}^{-u}\, L_m(u)L_n(u)L_l(u), \qquad c_{m,n,l} = c_{n,m,l} = c_{l,m,n}. \tag{7.6}$$

We do not know a practicable complete solution of this integration problem. However, we can successively determine the terms of the decomposition (7.4) beginning with the highest term corresponding to $l = m+n$ and then the term with $l = m+n-1$ from the remaining expression and so on. In this way, we obtained the following initial terms of the expansion in the order of decrease of the indices

$$\begin{aligned} L_m(u)L_n(u) \;&=\; \frac{(m+n)!}{m!n!} L_{m+n}(u) - \frac{2\,(m+n-1)!}{(m-1)!(n-1)!} L_{m+n-1}(u) \\ &+ \frac{(2mn-(m+n)+1)\,(m+n-2)!}{(m-1)!(n-1)!} L_{m+n-2}(u) \\ &- \frac{2\,(2mn-(m+n)+2)\,(m+n-3)!}{3\,(m-2)!(n-2)!} L_{m+n-3}(u) + \ldots . \end{aligned} \tag{7.7}$$

The lowest term corresponding to $l = 0$ is also clear from the integral in (7.6) which gives the Kronecker tensor $\delta_{m,n}$ for $l = 0$. With these initial terms, one can completely analyse simple cases with not too high m and (or) n and we found, for example

$$\begin{aligned} \mathrm{Conv}(|0\rangle\langle 0|, |n\rangle\langle n|) \;&=\; |n\rangle\langle n|, \\ \mathrm{Conv}(|1\rangle\langle 1|, |n\rangle\langle n|) \;&=\; (n+1)\,|n+1\rangle\langle n+1| - 2n\,|n\rangle\langle n| + n\,|n-1\rangle\langle n-1|, \\ \mathrm{Conv}(|2\rangle\langle 2|, |n\rangle\langle n|) \;&=\; \frac{(n+2)(n+1)}{2}\,|n+2\rangle\langle n+2| - 2(n+1)n\,|n+1\rangle\langle n+1| \\ &+(3n-1)n\,|n\rangle\langle n| - 2n(n-1)\,|n-1\rangle\langle n-1| \\ &+\frac{n(n-1)}{2}\,|n-2\rangle\langle n-2|, \end{aligned} \tag{7.8}$$

in particular

$$\begin{aligned}\mathrm{Conv}(|1\rangle\langle 1|, |1\rangle\langle 1|) &= 2\,|2\rangle\langle 2| - 2\,|1\rangle\langle 1| + |0\rangle\langle 0|, \\ \mathrm{Conv}(|1\rangle\langle 1|, |2\rangle\langle 2|) &= 3\,|3\rangle\langle 3| - 4\,|2\rangle\langle 2| + 2\,|1\rangle\langle 1|, \\ \mathrm{Conv}(|2\rangle\langle 2|, |2\rangle\langle 2|) &= 6\,|4\rangle\langle 4| - 12\,|3\rangle\langle 3| + 10\,|2\rangle\langle 2| - 4\,|1\rangle\langle 1| + |0\rangle\langle 0|. \quad (7.9)\end{aligned}$$

The right-hand sides in the last three examples show that the obtained density operators do not correspond to a positively definite Hermitean operator and that the "probabilities" in front of the projection operators are not restricted to be less than 1 in modulus in significant contradiction to the necessary requirements. Therefore, in case that one obtains a new possible Glauber-Sudarshan quasiprobability of a "physical" state by convolution of two such quasiprobabilities, there should be another, as a rule, more special mechanism which provides these results that we now illustrate.

We consider the combination of a coherent state $\varrho_1 = |\beta\rangle\langle\beta|$ with an arbitrary state $\varrho_2 = \varrho$ in the sense of (7.1) and obtain by convolution of its quasiprobabilities

$$P'(\alpha,\alpha^*) = \delta(\alpha-\beta,\alpha^*-\beta^*) * P(\alpha,\alpha^*) = P(\alpha-\beta,\alpha^*-\beta^*). \quad (7.10)$$

The new quasiprobability $P'(\alpha,\alpha^*)$ is therefore simply the old quasiprobability $P(\alpha,\alpha^*)$ displaced in the complex phase plane. Due to the "displacement structure" of the quasiprobabilities themselves [9], this is not only true for the Glauber-Sudarshan quasiprobability $P(\alpha,\alpha^*)$ but for all quasiprobabilities, for example, for $F_{(r_1,r_2,r_3)}(\alpha,\alpha^*)$ with arbitrary $\boldsymbol{r} = (r_1,r_2,r_3)$. The corresponding state can be constructed in this case by

$$\varrho' = D(\beta,\beta^*)\varrho\left(D(\beta,\beta^*)\right)^\dagger, \qquad D(\beta,\gamma) \equiv \exp\left(\beta a^\dagger - \gamma a\right), \quad (7.11)$$

where $D(\beta,\beta^*)$ denotes the unitary displacement operator. This means that in the considered special case the combination according to (7.1) is equivalent to the unitary transformation of the density operator ϱ according to (7.11) which is a legal state transformation in quantum theory. For example, the combination of a coherent state and of a thermal state in the considered sense leads to a new state which does not violate any fundamental assumptions of quantum theory and which plays a role in laser theory [5]. As discussed in [2,3], the inversion of the convolution theorem is sometimes useful to separate the state into components with simpler properties. The main example is again that we transform a state with a given quasiprobability $P'(\alpha,\alpha^*)$ and expectation value $\overline{a} = \langle\varrho' a\rangle$ in such a way that the new separated state ϱ with the displaced quasiprobability $P(\alpha,\alpha^*)$ has the expectation value $\overline{a} = \langle\varrho a\rangle = 0$ and we have separated then a coherent component from a remaining component with vanishing $\langle\varrho a\rangle$ by an inverse transformation of the kind (7.11).

Transformations of quasiprobabilities leading to quasiprobabilities of new "physical" states are of interest. Recently was shown [19] that the transformation of the coherent-state (Husimi) quasiprobability $Q(\alpha,\alpha^*) \to Q'(\alpha,\alpha^*) = |\mu|^2 Q(|\mu|\alpha, |\mu|\alpha^*)$ with $|\mu| < 1$ leads to a new possible coherent-state quasiprobability $Q'(\alpha,\alpha^*)$. The restriction to real $|\mu|$ can be released and one can write more generally $Q(\alpha,\alpha^*) \to Q'(\alpha,\alpha^*) = |\mu|^2 Q(\mu\alpha, \mu^*\alpha^*)$ with $|\mu| \leq 1$ including in this way rotations. The physical process which makes this transformation is the well-known phase-insensitive amplification [20,21]. In an analogous way, the transformation $P(\alpha,\alpha^*) \to P'(\alpha,\alpha^*) = |\nu|^2 P(\nu\alpha,\nu^*\alpha^*)$ with $|\nu| \geq 1$ leads to a new Glauber-Sudarshan quasiprobability $P'(\alpha,\alpha^*)$. The physical process is here absorption of the system in a reservoir with absolute temperature $T = 0$ [20,21]. For finite temperatures T correponding to a mean value $\overline{N} > 0$ of the harmonical oscillator, there is the quasiprobability $F_{(0,0,r)}(\alpha,\alpha^*)$ with $r = -(1+2\overline{N})$ which transforms under condition of absorption in a similar way. It is an "exotic" quasiprobability outside the sphere of quasiprobabilities $F_{\boldsymbol{r}}(\alpha,\alpha^*)$ restricted by the radius $\boldsymbol{r}^2 \leq 1$ as usually used in quantum optics and "more singular" as $P(\alpha,\alpha^*)$ but, nevertheless, it is a quasiprobability in full rights [9].

8. Conclusion

We have derived relations between different representations of the Glauber-Sudarshan quasiprobability in the Fock-state basis, in particular, the connection to the representation by the one-dimensional delta function given by Sudarshan. A modified representation of this kind involving Hermite polynomials was derived (Eq.(4.3)). The Peřina-Mišta representation as a regularized representation of the Glauber-Sudarshan quasiprobability was represented in a symmetric way by introduction of the Laguerre $2D$-functions. It was shown by explicit examples that the convolution of Glauber-Sudarshan quasiprobabilities does not lead, in general, to new Glauber-Sudarshan quasiprobabilities of "physical" states because it violates the positive definiteness of the corresponding density operator. Therefore, this convolution cannot be considered as a general combination principle. In the Appendices, we collect some formulas including derivations, in particular, for generalized functions of two variables in real and complex representation which are often useful in quantum optics.

Appendix A: Laguerre derivatives of Gaussian functions

We calculate in this Appendix polynomial derivatives of two-dimensional Gaussian functions which are important for the explicit calculation of (2.13) but play a role in many other problems.

By using Leibniz's rule for differentiation of products and by an obvious substitution of one summation index in the arising double sum and after reordering the double sum, one can calculate the inner sum and obtains the following chain of identities

$$
\begin{aligned}
& \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \frac{\partial^{m+n-2j}}{\partial z^{m-j} \partial z^{*\,n-j}} \exp\left(-\frac{zz^*}{\sigma}\right) \\
= \; & \exp\left(-\frac{zz^*}{\sigma}\right) \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \\
& \times \sum_{k=0}^{\{m-j,n-j\}} \frac{(m-j)!(n-j)!}{k!(m-j-k)!(n-j-k)!} \left(-\frac{1}{\sigma}\right)^{m+n-2j-k} z^{*\,m-j-k} z^{n-j-k}
\end{aligned}
$$

$$= \exp\left(-\frac{zz^*}{\sigma}\right)\left(-\frac{1}{\sigma}\right)^{m+n}\sum_{l=0}^{\{m,n\}}\frac{m!n!}{l!(m-l)!(n-l)!}\big(-\sigma(1-\sigma)\big)^l z^{*\,m-l}z^{n-l}. \tag{A.1}$$

Expressed by means of the Laguerre $2D$-functions, this takes on the form

$$\begin{aligned}&\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}\frac{\partial^{m+n-2j}}{\partial z^{m-j}\partial z^{*\,n-j}}\exp\left(-\frac{zz^*}{\sigma}\right)\\ &= \sqrt{\pi}\exp\left(\frac{(2\sigma-1)zz^*}{2\sigma(1-\sigma)}\right)\sqrt{m!n!}\left(-\sqrt{\frac{1-\sigma}{\sigma}}\right)^{m+n} l_{n,m}\left(\frac{z}{\sqrt{\sigma(1-\sigma)}},\frac{z^*}{\sqrt{\sigma(1-\sigma)}}\right).\end{aligned} \tag{A.2}$$

We mention here the following identities

$$\begin{aligned}\exp\left(-\tau\frac{\partial^2}{\partial z\partial z^*}\right)z^{*\,m}z^n &= (-\tau)^{m+n}\exp\left(\frac{zz^*}{\tau}\right)\frac{\partial^{m+n}}{\partial z^m\partial z^{*\,n}}\exp\left(-\frac{zz^*}{\tau}\right)\\ &= \sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}(-\tau)^j z^{*\,m-j}z^{n-j}\\ &= \sqrt{\pi}\exp\left(\frac{zz^*}{2\tau}\right)\sqrt{m!n!}\left(\sqrt{\tau}\right)^{m+n} l_{n,m}\left(\frac{z}{\sqrt{\tau}},\frac{z^*}{\sqrt{\tau}}\right),\end{aligned} \tag{A.3}$$

which can be verified by direct calculation and lead to modified definitions of the Laguerre $2D$-functions.

Appendix B: Moment series expansions

We derive in this Appendix the moment series expansion of functions and begin with functions of one real variable (see also Lukš [12]).

If we write a function $f(x)$ of the real variable x in form of a convolution with the delta function $\delta(x)$ according to

$$f(x) = \delta(x) * f(x) = \int_{-\infty}^{+\infty} dy\, f(y)\delta(x-y), \tag{B.1}$$

and if we make a Taylor series expansion of the delta function $\delta(x-y)$ in powers of y, we obtain

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_{-\infty}^{+\infty} dy\, f(y)y^n\right)\frac{\partial^n}{\partial x^n}\delta(x) \equiv \sum_{n=0}^{\infty} f_n\delta^{(n)}(x), \tag{B.2}$$

where f_n denotes the moments of the function $f(x)$ and $\delta^{(n)}(x)$ the n-th derivative of the delta function as follows

$$f_n = \frac{(-1)^n}{n!}\int_{-\infty}^{+\infty} dx f(x) x^n, \qquad \delta^{(n)}(x) \equiv \frac{\partial^n}{\partial x^n}\delta(x). \tag{B.3}$$

Expansions of the form (B.2) with the moments f_n of the function $f(x)$ in front we call moment series expansions (in analogy to Taylor series expansions). The reconstruction of the function $f(x)$ from this series has to be understood in the sense of weak convergence of generalized functions. Practically, one has to determine a space of basis functions $\varphi(x)$ in such a way that the partial sums of the linear functionals $\left(\sum_{n=0}^{\infty} f_n \delta^{(n)}(x), \varphi(x)\right)$ converge to the linear functional $(f(x), \varphi(x))$. It is inconvenient to discuss the necessary spaces in a physics paper and we make only some short remarks. One can suppose that in every case where all moments f_n exist, there can be found a space of basic functions $\varphi(x)$ in such a way that the moment series converges in the sense of weak convergence. For the existence of all moments of the function $f(x)$ it is necessary that this function rapidly decreases in infinity but it can be a generalized function with no restriction to the smoothness in arbitrary finite points. This is not one of the standard spaces $\mathcal{S}'$ of moderately increasing generalized functions (tempered distributions) or $\mathcal{D}'$ of arbitrarily increasing generalized functions. It is rather the subspace of rapidly decreasing generalized functions of the space $\mathcal{S}'$ of moderately increasing functions which we called the space $\mathcal{T}'$ in [22] (see also considerations in [5]). It can be determined as the space of continuous linear functionals over the space $\mathcal{T}$ of moderately increasing smooth (infinitely continuously differentiable) basis functions. We do not have here the usual inclusion relations of spaces of basis and of generalized functions as for the standard spaces $\mathcal{D} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{D}'$. For the existence of moment series expansions, the space of possible functions $\mathcal{T}'$ can be extended to a corresponding space $\mathcal{A}'$ of analytic linear functionals [22]. The space $\mathcal{A}'$ is a subspace of the space of analytic functionals $\mathcal{Z}'$ which is the space of Fourier transforms of generalized functions of the most common space of generalized functions $\mathcal{D}'$ [23–25] that means $\mathcal{A}' \subset \mathcal{Z}' \equiv \mathrm{F}(\mathcal{D}')$ [5] (scheme of inclusion relations in [22]).

After Fourier transformation according to

$$\tilde{f}(u) = \int_{-\infty}^{+\infty} dx f(x) \mathrm{e}^{-iux}, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \tilde{f}(u) \mathrm{e}^{iux}, \tag{B.4}$$

the moment series (B.3) takes on the form of a Taylor series of the Fourier transform $\tilde{f}(u)$ of $f(x)$

$$\tilde{f}(u) = \sum_{n=0}^{\infty} f_n (iu)^n, \qquad f_n = \frac{(-i)^n}{n!}\frac{\partial^n \tilde{f}}{\partial u^n}(0). \tag{B.5}$$

The reconstruction of a moment series can be also made via the reconstruction from the Taylor series of its Fourier transform and subsequent inversion of the Fourier transformation.

As an example, the moment series expansion of a normalized Gaussian function possesses the following form

$$\frac{1}{\sqrt{\pi}c}\exp\left(-\frac{x^2}{c^2}\right)=\exp\left(\frac{c^2}{4}\frac{\partial^2}{\partial x^2}\right)\delta(x)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{c}{2}\right)^{2m}\delta^{(2m)}(x). \tag{B.6}$$

Due to the normalization of the function, one has $f_0=1$ and due to the symmetry of the considered function, the moments f_n with odd n are vanishing. For a normalized displaced Gaussian function, one obtains

$$\begin{aligned}\frac{1}{\sqrt{\pi}c}\exp\left(-\frac{(x-x_0)^2}{c^2}\right) &= \exp\left(-x_0\frac{\partial}{\partial x}+\frac{c^2}{4}\frac{\partial^2}{\partial x^2}\right)\delta(x)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\frac{c}{2}\right)^n H_n\left(i\frac{x_0}{c}\right)\delta^{(n)}(x),\end{aligned} \tag{B.7}$$

where the expansion of the Hermite polynomials leads finally to real moments. Argument displacements can be similarly treated also in other cases.

It is not difficult to generalize the moment series to functions of several variables. For a function $f(x,y)$ of two Cartesian variables (x,y), one obtains

$$f(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{m,n}\frac{\partial^{m+n}}{\partial x^m\partial y^n}\delta(x)\delta(y),\qquad \tilde{f}(u,v)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{m,n}\,i^{m+n}u^m v^n, \tag{B.8}$$

with the Fourier transform and the moments defined by (we omit the integration regions which are the whole Euclidian spaces R^2 or dual Euclidian spaces $\tilde{\mathrm{R}}^2$)

$$\begin{aligned}\tilde{f}(u,v)&=\int dx\wedge dy\, f(x,y)\mathrm{e}^{-i(ux+vy)},\quad f(x,y)=\frac{1}{(2\pi)^2}\int du\wedge dv\,\tilde{f}(u,v)\mathrm{e}^{i(ux+vy)},\\ f_{m,n}&=\frac{(-1)^{m+n}}{m!n!}\int dx\wedge dy\, f(x,y)x^m y^n=\frac{(-i)^{m+n}}{m!n!}\frac{\partial^{m+n}\tilde{f}}{\partial u^m\partial v^n}(0,0).\end{aligned} \tag{B.9}$$

The complex representation of two-dimensional functions $f(z,z^*)$ can be obtained by the relations (for the transition $f(x,y)\to f(z,z^*)$, we do not invent a new function symbol different from f)

$$\begin{aligned}&z=x+iy,\quad z^*=x-iy,\quad \frac{\partial}{\partial x}=\frac{\partial}{\partial z}+\frac{\partial}{\partial z^*},\quad \frac{\partial}{\partial y}=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z^*}\right),\\ &w=u+iv,\quad w^*=u-iv,\quad \frac{\partial}{\partial u}=\frac{\partial}{\partial w}+\frac{\partial}{\partial w^*},\quad \frac{\partial}{\partial v}=i\left(\frac{\partial}{\partial w}-\frac{\partial}{\partial w^*}\right),\\ &u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}=w\frac{\partial}{\partial z}+w^*\frac{\partial}{\partial z^*},\quad dx\wedge dy=\frac{i}{2}dz\wedge dz^*,\quad du\wedge dv=\frac{i}{2}dw\wedge dw^*,\end{aligned} \tag{B.10}$$

and by the identification

$$\delta(z,z^*)\equiv\delta(x)\,\delta(y), \tag{B.11}$$

and takes on the following form (note that $f_{m,n}$ in the next formulas is different from $f_{m,n}$ in (B.8) and (B.9))

$$f(z,z^*) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\frac{\partial^{m+n}}{\partial z^m \partial z^{*\,n}}\delta(z,z^*), \quad \tilde{f}(w,w^*) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{m,n}\left(\frac{i}{2}\right)^{m+n} w^{*\,m}w^n, \tag{B.12}$$

with the Fourier transform and the moments defined by

$$\begin{aligned}
\tilde{f}(w,w^*) &= \int \frac{i}{2}dz\wedge dz^* f(z,z^*)\exp\left\{-\frac{i}{2}(w^*z+wz^*)\right\},\\
f(z,z^*) &= \frac{1}{(2\pi)^2}\int \frac{i}{2}dw\wedge dw^*\tilde{f}(w,w^*)\exp\left\{\frac{i}{2}(w^*z+wz^*)\right\},\\
f_{m,n} &= \frac{(-1)^{m+n}}{m!n!}\int \frac{i}{2}dz\wedge dz^* f(z,z^*)z^m z^{*\,n} = \frac{(-i2)^{m+n}}{m!n!}\frac{\partial^{m+n}\tilde{f}}{\partial w^{*\,m}\partial w^n}(0,0).
\end{aligned} \tag{B.13}$$

As an example, one obtains for a Gaussian function the following moment series

$$\frac{1}{\pi\sigma}\exp\left(-\frac{zz^*}{\sigma}\right) = \exp\left(\sigma\frac{\partial^2}{\partial z\partial z^*}\right)\delta(z,z^*) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}\frac{\partial^{2n}}{\partial z^n\partial z^{*\,n}}\delta(z,z^*), \tag{B.14}$$

This expansion is true, at least, for real non-negative σ where the normalized Gaussian function is rapidly vanishing in infinity.

Appendix C: Products of derivatives of delta functions with smooth functions

The multiplication of generalized functions with classes of well-behaved functions is defined in every monograph about generalized functions [23–25]. However, the specialization to the multiplication of derivatives of the delta function with smooth funtions (smooth, at least, at the singularities of the delta function) is rarely considered in general explicit form (in the monographs which we more or less studied [23–25]). We consider this here and derive the corresponding formulas.

We begin with the one-dimensional case. With multiplicator functions $g(x)$ and arbitrary basis functions $\varphi(x)$, one can make the following transformation of a linear functional

$$\begin{aligned}
\left(g(x)\,\delta^{(n)}(x),\varphi(x)\right) &= \left(\delta^{(n)}(x)g(x)\varphi(x)\right)\\
&= (-1)^n\left(\delta(x),\frac{\partial^n}{\partial x^n}\left(g(x)\varphi(x)\right)\right)\\
&= (-1)^n\sum_{l=0}^{n}\frac{n!}{l!(n-l)!}\left(\delta(x),g^{(l)}(x)\varphi^{(n-l)}(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n \sum_{l=0}^{\infty} \frac{n!}{l!(n-l)!} g^{(l)}(0) \varphi^{(n-l)}(0) \\
&= \sum_{l=0}^{n} \frac{(-1)^l n!}{l!(n-l)!} g^{(l)}(0) \left(\delta^{(n-l)}(x), \varphi(x) \right). \qquad \text{(C.1)}
\end{aligned}
$$

Since $\varphi(x)$ is an arbitrary basis function, this implies the identity

$$
g(x)\,\delta^{(n)}(x) = \sum_{l=0}^{n} \frac{(-1)^l n!}{l!(n-l)!} g^{(l)}(0)\,\delta^{(n-l)}(x). \qquad \text{(C.2)}
$$

This derivation can be easily generalized to functions of several variables with the following result in the two-dimensional case

$$
\begin{aligned}
g(x,y) \frac{\partial^{m+n}}{\partial x^m \partial y^n} \delta(x)\delta(y) &= \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-1)^{k+l} m! n!}{k!(m-k)! l!(n-l)!} \frac{\partial^{k+l} g}{\partial x^k \partial y^l}(0,0) \\
&\quad \times \frac{\partial^{m+n-k-l}}{\partial x^{m-k} \partial y^{n-l}} \delta(x)\delta(y). \qquad \text{(C.3)}
\end{aligned}
$$

The corresponding identity in complex representation takes on the form

$$
\begin{aligned}
g(z,z^*) \frac{\partial^{m+n}}{\partial z^m \partial z^{*\,n}} \delta(z,z^*) &= \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{(-1)^{k+l} m! n!}{k!(m-k)! l!(n-l)!} \frac{\partial^{k+l} g}{\partial z^k \partial z^{*\,l}}(0,0) \\
&\quad \times \frac{\partial^{m+n-k-l}}{\partial z^{m-k} \partial z^{*\,n-l}} \delta(z,z^*). \qquad \text{(C.4)}
\end{aligned}
$$

As a first example for the one-dimensional case, we consider $g(x) = x^l$ and find from (C.2)

$$
x^l\,\delta^{(n)}(x) = \frac{(-1)^l n!}{(n-l)!} \delta^{(n-l)}(x) = \frac{(-1)^k n!}{(n-k)!} x^{l-k} \delta^{(n-k)}(x), \qquad k = 0, 1, \ldots, l. \qquad \text{(C.5)}
$$

As a second example, we consider Gaussian functions $g(x) = \exp(-x^2/c^2)$ and obtain from (C.2)

$$
\exp\left(-\frac{x^2}{c^2}\right) \delta^{(n)}(x) = \sum_{j=0}^{[\frac{n}{2}]} \frac{(-1)^j n!}{j!(n-2j)!} \left(\frac{1}{c}\right)^{2j} \delta^{(n-2j)}(x) = \frac{1}{c^n} H_n\left(\frac{c}{2}\frac{\partial}{\partial x}\right) \delta(x), \qquad \text{(C.6)}
$$

with $H_n(z)$ as the Hermite polynomials. As a first example for the two-dimensional case in complex representation, we consider $g(z,z^*) = z^k z^{*\,l}$ and obtain from (C.4)

$$
z^k z^{*\,l} \frac{\partial^{m+n}}{\partial z^m \partial z^{*\,n}} \delta(z,z^*) = \frac{(-1)^{k+l} m! n!}{(m-k)!(n-l)!} \frac{\partial^{m+n-k-l}}{\partial z^{m-k} \partial z^{*\,n-l}} \delta(z,z^*). \qquad \text{(C.7)}
$$

As a second example for the two-dimensional case, we consider the function $g(z,z^*) = \exp(-zz^*/\sigma)$. By direct calculation, one finds from (C.4)

$$\begin{aligned} &\exp\left(-\frac{zz^*}{\sigma}\right)\frac{\partial^{m+n}}{\partial z^m \partial z^{*\,n}}\delta(z,z^*) \\ = &\sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!}\left(-\frac{1}{\sigma}\right)^j \frac{\partial^{m+n-2j}}{\partial z^{m-j}\partial z^{*\,n-j}}\delta(z,z^*) \\ = &\; n!\left(-\frac{1}{\sigma}\right)^n \frac{\partial^{m-n}}{\partial z^{m-n}} L_n^{m-n}\left(\sigma\frac{\partial^2}{\partial z \partial z^*}\right)\delta(z,z^*). \end{aligned} \tag{C.8}$$

One can look to the formulas of this Appendix as to the "disentanglement" of products of functions with delta functions and their derivatives.

Appendix D: Representation of two-dimensional delta function by central-symmetric one-dimensional delta functions

The N–dimensional delta function $\delta^N(\boldsymbol{r})$ over an N–dimensional Euclidian space can be obtained by applying the Laplacean $\boldsymbol{\nabla}^2$ to a central-symmetric function as follows [23,24]

$$\begin{aligned} &\delta^N(\boldsymbol{r}) = \boldsymbol{\nabla}^2\left(-\frac{1}{S_N(N-2)|\boldsymbol{r}|^{N-2}}\right), \quad N \neq 2, \quad S_N = \oint_{|\boldsymbol{r}|=1} d^{N-1}n = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \\ &\delta^2(\boldsymbol{r}) = \boldsymbol{\nabla}^2\left(\frac{\log|\boldsymbol{r}|}{S_2}\right), \quad \log|\boldsymbol{r}| = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(1-\frac{1}{|\boldsymbol{r}|^{\varepsilon}}\right), \quad S_2 = 2\pi, \quad \boldsymbol{n} \equiv \frac{\boldsymbol{r}}{|\boldsymbol{r}|}, \end{aligned} \tag{D.1}$$

where S_N is the (hyper-) area of the N–dimensional unit hypersphere. This relation shows that the N–dimensional delta function can be considered as a central-symmetric generalized function which depends only on $|\boldsymbol{r}|$ but not on the coordinates on the unit sphere. We can substitute (D.1) by the following equation in the sense of a limiting procedure

$$\begin{aligned} \delta^N(\boldsymbol{r}) &= \lim_{\varepsilon\to 0}\boldsymbol{\nabla}^2\left(-\frac{\Theta(|\boldsymbol{r}|-\varepsilon)}{S_N(N-2)|\boldsymbol{r}|^{N-2}}\right) \\ &= \lim_{\varepsilon\to 0}\left(\frac{\partial^2}{(\partial|\boldsymbol{r}|)^2} + \frac{N-1}{|\boldsymbol{r}|}\frac{\partial}{\partial|\boldsymbol{r}|}\right)\left(-\frac{\Theta(|\boldsymbol{r}|-\varepsilon)}{S_N(N-2)|\boldsymbol{r}|^{N-2}}\right) \\ &= \lim_{\varepsilon\to 0}\frac{\delta(|\boldsymbol{r}|-\varepsilon)}{S_N|\boldsymbol{r}|^{N-1}}, \end{aligned} \tag{D.2}$$

where $\delta(x)$ denotes the one-dimensional delta function and $\Theta(x)$ the Heaviside step function ($\Theta^{(1)}(x) = \delta(x)$). We used here the relation $|\boldsymbol{r}|\delta^{(1)}(|\boldsymbol{r}|-\varepsilon) = \varepsilon\,\delta^{(1)}(|\boldsymbol{r}|-\varepsilon) - \delta(|\boldsymbol{r}|-\varepsilon)$ which can be obtained from the multiplication of delta functions and their derivatives with smooth functions (see Appendix C). The N–dimensional delta function is in (D.2) substituted by a one-dimensional delta function combined with the factor $|\boldsymbol{r}|^{N-1}$ in the denominator which is absorbed in N–dimensional integrations and by a limiting procedure. This is in the sense

$$\int d^N r\, \delta^N(\boldsymbol{r})\varphi(\boldsymbol{r}) = \lim_{\varepsilon\to 0}\int_0^{+\infty} d|\boldsymbol{r}|\, |\boldsymbol{r}|^{N-1}\frac{\delta(|\boldsymbol{r}|-\varepsilon)}{|\boldsymbol{r}|^{N-1}}\frac{1}{S_N}\oint d^{N-1}n\,\varphi(|\boldsymbol{r}|\boldsymbol{n}) = \varphi(\boldsymbol{0}). \tag{D.3}$$

It seems to be not consequent if one substitutes the limiting procedure by adding a factor $1/2$ in front of the one-dimensional delta function. We see from (D.3) that it is allowed to have a factor $1/|\boldsymbol{r}|^{N-1}$ in front of the "one-dimensional" delta function $\delta(|\boldsymbol{r}|)$ or $\delta(|\boldsymbol{r}|-\varepsilon)$ if it is used in N-dimensional volume integrations because then this factor is absorbed by the volume element in spherical coordinates.

We now consider the two-dimensional case. It is clear that the result in (D.2) is also true for the two-dimensional case which, in principle, has to be treated separately. In the two-dimensional case, one can write in real and complex representation

$$\delta^2(\boldsymbol{r}) = \lim_{\varepsilon\to 0}\frac{\delta(|\boldsymbol{r}|-\varepsilon)}{2\pi|\boldsymbol{r}|} = \lim_{\varepsilon\to 0}\frac{\delta(|z|-\varepsilon)}{2\pi|z|} = \delta(z,z^*). \tag{D.4}$$

We now derive the representation of the derivatives of the two-dimensional delta function by the one-dimensional central-symmetric delta function and use the complex representation. In this representation we set

$$z = |z|\mathrm{e}^{i\varphi},\;\; z^* = |z|\mathrm{e}^{-i\varphi},\;\; \frac{\partial}{\partial z} = \frac{\mathrm{e}^{-i\varphi}}{2}\left(\frac{\partial}{\partial|z|} - \frac{i}{|z|}\frac{\partial}{\partial\varphi}\right),\;\; \frac{\partial}{\partial z^*} = \frac{\mathrm{e}^{i\varphi}}{2}\left(\frac{\partial}{\partial|z|} + \frac{i}{|z|}\frac{\partial}{\partial\varphi}\right). \tag{D.5}$$

The derivatives of the two-dimensional delta function can be substituted in the following way by using the one-dimensional central-symmetric delta function

$$\frac{\partial^{m+n}}{\partial z^m\partial z^{*\,n}}\delta(z,z^*) = \frac{m!n!}{(m+n)!}\frac{\mathrm{e}^{i(n-m)\varphi}}{2\pi|z|}\lim_{\varepsilon\to 0}\frac{\partial^{m+n}}{\partial|z|^{m+n}}\delta(|z|-\varepsilon). \tag{D.6}$$

This can be proved by complete induction $m\to m+1$ and $n\to n+1$ by using (D.5) and by using the multiplication of the one-dimensional delta function $\delta(|z|-\varepsilon)$ with powers of $|z|$ (see Appendix C). For example

$$\begin{aligned}
&\frac{\partial^{m+1+n}}{\partial z^{m+1}\partial z^{*\,n}}\delta(z,z^*)\\
= \;& \frac{\mathrm{e}^{-i\varphi}}{2|z|}\left(\frac{\partial}{\partial|z|}|z| - 1 - i\frac{\partial}{\partial\varphi}\right)\frac{m!n!}{(m+n)!}\frac{\mathrm{e}^{i(n-m)\varphi}}{2\pi|z|}\lim_{\varepsilon\to 0}\delta^{(m+n)}(|z|-\varepsilon)\\
= \;& \frac{m!n!}{(m+n)!}\frac{\mathrm{e}^{i(n-m-1)\varphi}}{4\pi|z|}\lim_{\varepsilon\to 0}\left\{\delta^{(m+1+n)}(|z|-\varepsilon) - \frac{m+1-n}{|z|}\delta^{(m+n)}(|z|-\varepsilon)\right\}\\
= \;& \frac{m!n!}{(m+n)!}\frac{\mathrm{e}^{i(n-m-1)\varphi}}{4\pi|z|}\lim_{\varepsilon\to 0}\left\{\left(1+\frac{m+1-n}{m+1+n}\right)\delta^{(m+1+n)}(|z|-\varepsilon)\right\}\\
= \;& \frac{(m+1)!n!}{(m+1+n)!}\frac{\mathrm{e}^{i(n-m-1)\varphi}}{2\pi|z|}\lim_{\varepsilon\to 0}\delta^{(m+1+n)}(|z|-\varepsilon),
\end{aligned} \tag{D.7}$$

where is used

$$|z|\,\delta^{(m+n+1)}(|z|-\varepsilon)=\varepsilon\,\delta^{(m+n+1)}(|z|-\varepsilon)-(m+n+1)\,\delta^{(m+n)}(|z|-\varepsilon), \qquad \text{(D.8)}$$

which can be divided by $|z|$ for $\varepsilon \neq 0$, and analogously the proof for $n \to n+1$.

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