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VALIDITY OF THE CUMULANT METHOD FOR A PULSE NONLINEAR KERR OSCILLATOR [1]

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We study the dynamics of an anharmonic oscillator driven by a train of pulses. The cumulant expansion and quantum evolution operator approaches are presented and compared.

1. Introduction

The modifications introduced by quantum mechanics into the dynamics of classical systems which manifest chaos are a problem of great importance [1]. It is known that quantization modifies the dynamics of classical chaotic systems [2,3,4].

The time evolution of a quantum optical system is usually studied by means of the equation for the Wigner function derived from the quantum Liouville equation. In Wigner's formulation of quantum mechanics we treat a quantum system in a "classical way" including all their quantum features. And what is more, we can contrast the quantum and classical dynamics within the framework of one formalism [3-6].

The problem is, that the equations for the Wigner functions are mathematically cumbersome and their analytic solutions for most nonlinear systems are unknown. However, instead of the equation for the Wigner function we can use the set of equations for statistical moments generated by our equation for the Wigner function. It is obvious that in this approach a quantum system is governed by an infinite set of equations. Therefore, for numerical reasons the set of equations for statistical moments has to be truncated at a finite number, which means approximating it. It is known that the first cumulant approximation represents the classical dynamics. The second cumulant approximation adds the first quantum corrections to the classical dynamics.

In this paper we compare some aspects of the cumulant method and the method used by Leoński and Tanaś to study an anharmonic oscillator driven by a train of pulses. The Kerr oscillator model is the same as that is discussed in an earlier paper [3] albeit without the damping mechanism.

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2. The Model

We consider the Hamiltonian for the coupled *external field - anharmonic oscillator* in the form $\hat{H} = \hat{H}_1 + \hat{H}_2$, where

$$\hat{H}_1 = \frac{\chi}{2}\,\hat{a}^{\dagger 2}\hat{a}^2 \,, \tag{1}$$

$$\hat{H}_2 = F(t)\left(\hat{a}^\dagger + \hat{a}\right) \,. \tag{2}$$

The anharmonic oscillator model (describing the optical Kerr effect) is described by the single mode Hamiltonian (1), where χ is the anharmonicity parameter. The quantities $\hat{a}$ and $\hat{a}^\dagger$ are the boson annihilation and creation operators, respectively. We use units $\hbar = 1$. The interaction between the classical external driving field $F(t)$ and the single-mode field is governed by the Hamiltonian (2). For the sake of simplicity the damping mechanism is not considered in the presented model.

The dynamics of our system (1)-(2) is studied in the standard Louisell approach [7]. At first, we obtain a master equation for the density operator. Then the master equation is transformed to a c-number partial differential equation for the Wigner quasidistribution function defined in the complex space ($\alpha = p + iq, \alpha^* = p - iq$) instead of the traditional (p,q) space. Our equation for the Wigner function W has the form of a generalized Fokker-Planck equation [3]:

$$\frac{\partial W}{\partial \tau} = K_{class} + K_{quant} \,, \tag{3}$$

where

$$K_{class} = \left[\frac{\partial}{\partial \alpha}\left(-i\mathcal{F}(t) + i\alpha \mid \alpha \mid^2\right) + \frac{\partial}{\partial \alpha^*}\left(i\mathcal{F}(t) - i\alpha^* \mid \alpha \mid^2\right)\right] W \,, \tag{4}$$

$$K_{quant} = -i\left[\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^* - \frac{1}{4}\frac{\partial^3}{\partial \alpha \partial \alpha^{*2}}\alpha^* + \frac{1}{4}\frac{\partial^3}{\partial \alpha^* \partial \alpha^2}\alpha\right] W \,. \tag{5}$$

We used a redefined time $\tau = \chi t$, and the external force $\mathcal{F}(t) = F(t)/\chi$. The dynamics of the system described by Eqs.(5) is studied for the case when $\mathcal{F}(t)$ is a train of rectangular pulses. The length of the pulse is denoted by T_1, whereas T_2 is the distance between pulses, and $\mathcal{F}_0$ is their height. In the classical limit Eq.(3) does not contain the term K_{quant} and W is a classical distribution function. In other words, the K_{quant}–term conveys the quantum (operator) correction to the classical description [3].

3. The cumulant method

The equation for the Wigner quasiprobability (3) generates an infinite hierarchic set of equations for the statistical moments and the other way round. This one to one relation is statistically strict as long as the set of equations for the statistical moments is infinite. However, for numerical reasons, the set of equations has to be truncated at a finite number, which means it is approximated. Integration *per partes* of Eq.(3) allows us to write the appropriate equations for the cumulants. This method, extensively studied earlier [3], leads to the following results. The first truncation (total factorization of the statistical moments $< a^n >=< a >^n$) leads to classical equations of motion which, in the autonomized version, have the form

$$\begin{aligned} \frac{d\xi}{d\tau} &= -i\mathcal{F}(w) - i\xi^2\xi^* \ , \\ \frac{dw}{d\tau} &= 1 \, , \ \ w(0) = 0 \ , \end{aligned} \tag{6}$$

where $\xi =< \hat{a} >$. As seen, the time τ in the amplitude $\mathcal{F}(t)$ has been deliberately changed into w to avoid the explicit dependence on the independent variable τ. The quantum term K_{quant} in (3) contributes nothing to the above set of equations. The set consists of three equations of motion in real variables $Re\xi$, $Im\xi$ and w, which means that if $\mathcal{F}(w) = const$ the system is two-dimensional and consequently, nonchaotic. If we restrict ourselves to the second truncation (Gaussian approximation) we arrive at the following set of equations:

$$\begin{aligned} \frac{d\xi}{d\tau} &= -i\mathcal{F}(w) - i[2B\xi + C\xi^* + \xi^2\xi^*] \, , \\ \frac{dw}{d\tau} &= 1 \, , \ \ w(0) = 0 \, , \\ \frac{dC}{d\tau} &= -i\left[\xi^2(1+2B) + C(1+4 \mid \xi \mid^2)\right] - 6iBC \, , \\ \frac{dB}{d\tau} &= i\left[C\xi^{*2} - C^*\xi^2\right] \ , \end{aligned} \tag{7}$$

where $B =< \hat{a}^\dagger\hat{a} > - < \hat{a}^\dagger >< \hat{a} >$ and $C =< \hat{a}^2 > - < \hat{a} >^2$. The above set (7) consists of six equations in the real variables $Re\xi$, $Im\xi$, ReC, ImC, B and w. The initial conditions have the form $\xi(0) = \xi_0$ and $B(0) = C(0) = 0$. The physical meaning of the second truncation is clear if we note (details in [3]) that the quantum term K_{quant} in (3) now adds nonzero corrections to the first truncation, that is, to the classical equations (6). The higher order truncations lead to an increasing number of equations and in consequence to more and more rigorous quantum corrections but they are relatively small. For example the correction caused by the third truncation is only of the rank 0.2% of the correction caused by the second truncation [3] .

4. Pure quantum description

Contrary to the previous section, where the classical model with quantum corrections has been discussed, this part of the paper is devoted to the pure quantum description of our model. Our method is based on the procedure discussed in [8,9]. We neglect damping processes and therefore, we describe the system using the standard wave-function approach. Moreover, since the external excitation is assumed to be rectangular, we shall divide the whole evolution into two stages. The first concerns the time between two subsequent pulses, when our system evolves as the usual nonlinear oscillator. In consequence, its behavior is governed by the Hamiltonian

$$\hat{H}_n = \frac{\chi}{2}\hat{a}^{\dagger 2}\hat{a}^2 \ . \tag{8}$$

As the external field starts acting on the system, the Hamiltonian becomes

$$\hat{H}_k = \frac{\chi}{2}\hat{a}^{\dagger 2}\hat{a}^2 + F_o(\hat{a}^\dagger + \hat{a}) \ . \tag{9}$$

Now, we are in a position to introduce unitary evolution operators. These operators are based on the Hamiltonians $\hat{H}_n$ and $\hat{H}_k$ and can be expressed as:

$$\hat{U}_n \quad = \quad e^{-i\chi\hat{n}(\hat{n}-1)T_1/2} \ , \tag{10}$$

and

$$\hat{U}_k \quad = \quad e^{-i(\chi\hat{n}(\hat{n}-1)/2+F_o(\hat{a}^\dagger+\hat{a}))T_2} \ , \tag{11}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the photon number operator and the parameters T_1 and T_2 are the time between two subsequent pulses and the pulse duration time, respectively. In this paper we are interested in the stroboscopic map of the evolution of our system for the moments corresponding to the times just after pulses. Therefore, we define the evolution operator transforming the wave-function describing our system for the time just after k-th pulse to that corresponding to the time after $(k+1)$-th pulse.

$$\hat{U} = \hat{U}_k\hat{U}_n \ . \tag{12}$$

Thus, assuming that the system was initially in the state $|\Psi(t=0)>$, we are able to determine the state of the system for all of the times after the an arbitrary pulse. The wave-functions corresponding to those times are defined by:

$$|\Psi_k\rangle = \hat{U}^k\,|\Psi(t=0)\rangle \ . \tag{13}$$

Although, this function enables us to determine various quantum properties of the system, we shall concentrate in this paper on the time-evolution of the mean number of photons $n =< \hat{n} >$. It can be easily calculated and is given by:

$$< \hat{n} >= \langle\Psi_k|\,\hat{a}^\dagger\hat{a}\,|\Psi_k\rangle \ . \tag{14}$$

We can perform the above calculations numerically and compare the results obtained using this method with those corresponding to the cumulant expansion technique, discussed in the previous section. To get the compact numerical results for both methods we have put $\chi = 1$. Therefore $\tau = w = t$, $F_o = \mathcal{F}_o$.

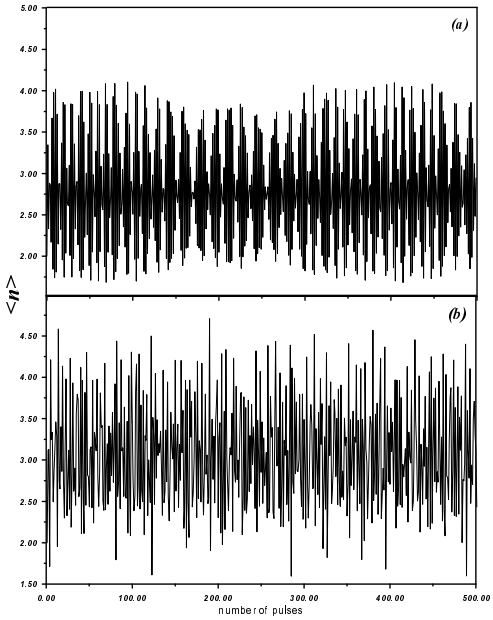

Fig. 1. The mean number of photons $< \hat{a}^\dagger \hat{a} >$ *versus* the number of pulses for the cumulant method (a) and quantum mapping (b) for the parameters of pulse: $F_o = 2.0$, $T_1 = 1.0$, $T_2 = 1.0$ and the initial conditions: $\xi_o = 1 + 1i$ and $B(0) = C(0) = 0$.

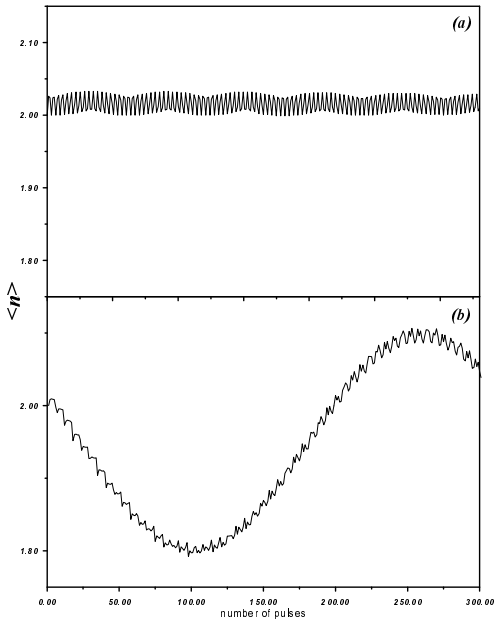

Fig. 2. The same as in Fig. 1, but for $F_o = 0.1$.

5. Numerical results

We observe a difference in the dynamics of our model studied with the help of the cumulant approximation and the pure quantum mapping. To solve the set of equations for the cumulants (7) we employed the standard fourth-order Runge-Kutta procedure. The pure quantum calculations is based on a discrete mapping of the quantum states. Both computer programs (for $F(t) = 0$) can be tested by printing the energy as a constant of motion.

In Fig. 1 we observe the average number of photons $\hat{n} = \hat{a}^\dagger \hat{a}$ versus the time t in the cumulant approximation (a) and the quantum mapping (b). Both oscillations are chaotic albeit a slight different in nature.

This is caused by the fact that the system is pumped without damping and in consequence it is strongly unstable (mostly chaotic). Moreover, the cumulant correction caused by the third and higher truncations are neglected as well as the evolution operator in the quantum description is calculated with numerical approximation method. However, as seen from Fig. 1 both methods give quantitatively similar results. The situation in Fig. 1 changes for weaker external field $F(t)$. It is seen in Fig. 2 for $F_0 = 0.1$.

The pure quantum procedure (Fig. 2b) gives rapid oscillations which modulate a slowly varying sine-like function whereas in the cumulant method the slowly varying function has a very small amplitude (Fig. 2a). The differences in Fig. 2 suggests that if the external field $F(t)$ is small relative to the initial number of photons the semiclassical cumulant approach is not so effective.

Both presented methods have their advantages and disadvantages. In our pulse case the cumulant method in the contradistinction to the pure quantum method can be used also when a damping mechanism is taken into account. On the other hand, the cumulant method is a kind of semiclassical approach and can be applied carefully if quantum evolution is dominant (Fig. 2). The difficulties in comparing the two methods are also due to the fact that the quantum mapping in the pulse case has no classical counterpart.

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