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PERIODIC BEHAVIOUR OF DISPLACED KERR STATES[1]

W. Leoński[2]
Nonlinear Optics Division, Institute of Physics, A. Mickiewicz University, ul. Umultowska 85, 61-614 Poznań, Poland

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We discuss quantum properties of displaced Kerr states, in particular the periodic behaviour of the mean values of various quantum parameters describing our model. Thus, we introduce an operator evolution approach that justifies our conclusions concerning the periodic behaviour of the system.

1. Introduction

Quantum systems that involve nonlinear Kerr-like media were the subject of numerous papers (see e.g. [1-16] *and references quoted therein*). Those systems were discussed from various points of view. Thus, models involving Kerr media can, under certain conditions, lead to the generation of various quantum states of the field. For instance, such systems can lead to n-photon states [3,7,9,10] or to superpositions of coherent states (Schrödinger cats) [3] generation. Moreover, systems with Kerr media can exhibit strong squeezing properties [1,2,5,14]. Those models can also be used as examples of systems exhibiting chaotic behaviour [5,11,15,16].

This paper is devoted to periodical properties of displaced Kerr states (DKS). Such states were discussed by Wilson-Gordon *et al.* [5], who proposed a system based on a Mach-Zender interferometer with a nonlinear medium in one of its arms. This system was irradiated by a coherent field producing in one of the interferometer output beams a state that has been referred to as displaced Kerr state. For such a state they derived an analytical solution for the expansion of the wave-function in n-photon basis. Using this result they investigated thoroughly various quantum properties of the DKS. For instance, they discussed the evolution of the Mandel Q parameter [17], quasi probability Q distribution, and squeezing of the field quadratures. Moreover, the phase properties of the system were also discussed .

[1] Special Issue on Quantum Optics and Quantum Information
[2] E-mail address: wleonski@main.amu.edu.pl

2. Displaced Kerr states – analytical solutions

The displaced Kerr states can be generated from the usual coherent state $|\alpha>$ by application of two operators. One of them is the unitary operator $\hat{U}_n$

$$\hat{U}_n = e^{i\frac{\chi}{2}\hat{n}(\hat{n}-1)} \quad , \tag{1}$$

where the parameter χ is a nonlinearity constant corresponding to the Kerr medium third-order susceptibility and $\hat{n}$ is the photon number operator. The evolution operator can be related to the Hamiltonian of the nonlinear oscillator $\hat{H}_n$ (in units of $\hbar = 1$):

$$\hat{H}_n = \chi(\hat{a}^\dagger)^2\hat{a}^2 \quad . \tag{2}$$

The nonlinear constant χ appearing in Eq.(1) can be treated as generalized time. In consequence, the system can be investigated as one evolving in time with nonlinearity constant equal to 1. By action of the operator $\hat{U}_n$ on the coherent state $|\alpha>$ we get the Kerr state $|\Psi_{Kerr}>$. This can be expressed as in [5]:

$$\begin{aligned} |\Psi_{Kerr}\rangle &= \hat{U}_n\,|\alpha\rangle \\ &= \exp(-1/2|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{n!^{1/2}}e^{i\frac{\chi}{2}n(n-1)}\,|n\rangle \quad . \end{aligned} \tag{3}$$

It is easily seen that this state differs from the usual coherent state by the presence of the phase factor $\exp(i\chi n(n-1)/2)$.

In the next step we transform the Kerr state to the DKS using the unitary evolution operator $\hat{U}_k$. This operator is the usual displacement operator and has the following form:

$$\hat{U}_k = e^{\xi\hat{a}^\dagger - \xi^*\hat{a}} \tag{4}$$

and conforms to the coherent excitation of the physical system corresponding to our model. In consequence, we obtain a state that is referred to as DKS

$$|\Psi_{DKerr}\rangle = \hat{U}_k\hat{U}_n\,|\alpha\rangle \quad . \tag{5}$$

It is possible to find an expansion of this state in n-photon basis. This has already been done by Wilson-Gordon *et al.* [5]. The coefficients of that expansion have a rather complicated form and our expansion can be expressed as:

$$|\Psi_{DKerr}\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle \quad , \tag{6}$$

where

$$\begin{aligned} c_n &= \exp(-(|\alpha|^2+|\xi|^2)/2)\sum_{m=0}^{\infty} e^{i\phi_\alpha m}e^{\frac{\chi}{2}m(m-1)}|\alpha|^m \\ &\times \sum_{k=0}^{\min[m,n]}(-1)^{m-k}\frac{n!^{1/2}}{k!(m-k)!(n-k)!}e^{i\phi_\xi(n-m)}|\xi|^{(n+m-2k)} \quad . \end{aligned} \tag{7}$$

The complex parameters α and ξ have been expressed as $\alpha = |\alpha|\exp(i\phi_\alpha)$ and $\xi = |\xi|\exp(i\phi_\xi)$, respectively (for simplicity, we shall assume that the parameters α and ξ are real). This formula (7) enables us to find the mean values of various operators. For instance, we can calculate the mean values of the annihilation operator $<\hat{a}>$, the squared annihilation operator $<\hat{a}^2>$ or the mean number of photons $<\hat{n}>$. They are given by:

$$
\begin{aligned}
<\hat{a}> &= \sum_{n=0}^{\infty} \sqrt{n+1} c_n^* c_{n+1} \; ; \\
<\hat{a}^2> &= \sum_{n=0}^{\infty} \sqrt{(n+1)(n+2)} c_n^* c_{n+2} \; ; \\
<\hat{n}> &= \sum_{n=0}^{\infty} n c_n^* c_n \; .
\end{aligned}
\tag{8}
$$

However, due to the rather complicated form of the coefficient c_n, it is difficult to find a compact form of the expressions describing mean values of the operators and hence, we cannot easily draw conclusions concerning the periodic behaviour of the system. Therefore, we propose an alternative method and instead of the transformations of the wave function we evolve the annihilation operator $\hat{a}$. This method has been applied in a paper by Gerry and Grobe [18] where the squeezed Kerr states were discussed. Assuming that during the evolution given by the unitary operator $\hat{U}_n$ the mean number of photons is preserved, the annihilation operator $\hat{a}$ is transformed to the following form:

$$
\hat{a}_{DKerr} = e^{-i\chi\hat{n}}\hat{a} + \xi \; , \tag{9}
$$

where the operator $\hat{a}_{DKerr}$ corresponds to the DKS.

Since we are interested in the quantum evolution of the parameters describing the system rather than the operators, assuming that the field was initially in the coherent state we find the mean value of the annihilation operator corresponding to the DKS. After some straightforward algebra we can write the mean value as :

$$
\langle \Psi_{DKerr}| \, \hat{a} \, |\Psi_{DKerr}\rangle = \langle \alpha| \, \hat{a}_{DKerr} \, |\alpha\rangle = \exp\left[-|\alpha|^2(1-e^{i\chi})\right]\alpha + \xi \; . \tag{10}
$$

Similarly, we can perform analogous calculations for various combinations of the operators $\hat{a}$ and $\hat{a}^\dagger$.

3. Numerical results and discussion

We are now in a position to calculate the Mandel Q parameter for various values of the nonlinear constant χ (one should keep in mind that the parameter χ can be treated as generalized time and therefore, we treat our system as evolving in time). The Q-parameter is defined as :

$$
Q = \frac{<\Delta\hat{n}^2> - <\hat{n}>}{<\hat{n}>} \; . \tag{11}
$$

Thus, Fig. 1 shows Q as a function of χ. We assume that the initial state is a coherent state $|\alpha\rangle$ with $\alpha = 4$ and the displacement parameter $\xi = 2$. This plot is identical to that shown in the paper [5]. We see that, similarly as in [5], the system exhibits sub-Poissonian statistics for $\chi < 0.13$ and super-Poissonian for $\chi > 0.13$. However, when we extend the time-scale (Fig. 2) – the remaining parameters are the same as for the case shown in Fig. 1 – the situation observed changes considerably. It is seen that the value of Q starts to oscillate and these oscillations are heavily damped. In consequence, Q reaches ~ 6.5 and remains constant. For $\chi \sim 2.8$ new oscillations become visible. After several oscillations they are damped and the Mandel Q-parameter becomes constant again. Next, we observe oscillations starting for $\chi \sim 5.5$ that are symmetrical to those for $\chi \epsilon < 0, 1 >$. In consequence, the behaviour of the Q-parameter resembles collapses and revivals. However, the essential feature of this evolution resides in its periodicity. The periodicity is visible as χ reaches 2π, when the Mandel parameter reaches the same value as for $\chi = 0$. Then Q starts to evolve identically as for lower values of χ.

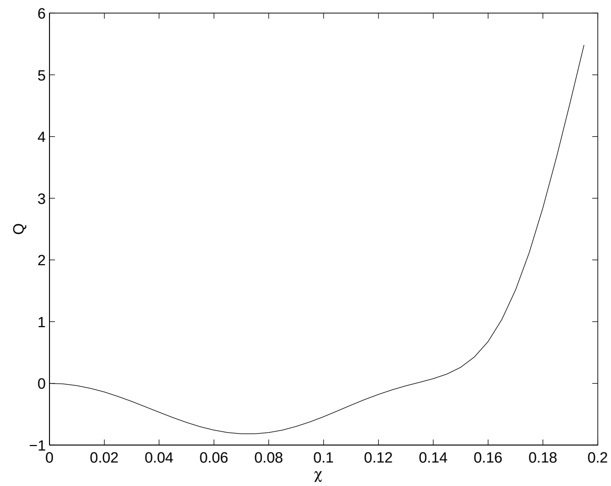

Fig. 1. Mandel Q-parameter for various values of χ. The parameters α and ξ are assumed to be real – $\alpha = 4$, $\xi = 2$.

A similar character of the evolution can be observed for the parameters describing squeezing. Thus, similarly as in [5], we can define quadrature operators $\hat{X}_1$ and $\hat{X}_2$:

$$\begin{aligned} \hat{X}_1 &= \frac{\hat{a} + \hat{a}^\dagger}{2} ; \\ \hat{X}_2 &= \frac{\hat{a} - \hat{a}^\dagger}{2i} \end{aligned} \tag{12}$$

and in consequence, squeezing parameters $S_{1,2}$:

$$S_{1,2} = 4 \left[\left(\langle \Delta \hat{a}_{1,2} \rangle^2 \right\rangle - 1/4 \right] . \tag{13}$$

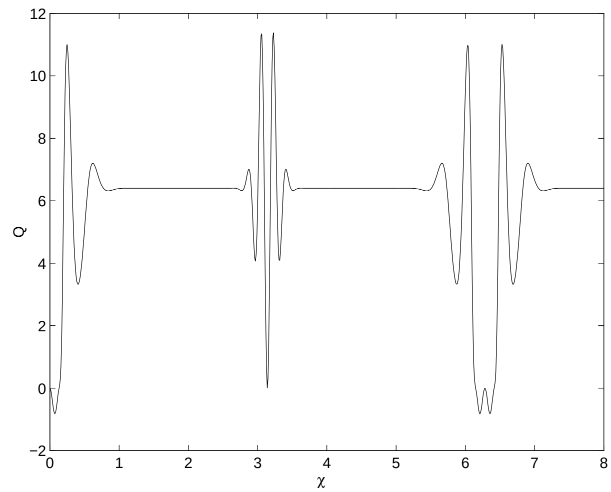

Fig. 2. The same as for Fig. 1, albeit for $\chi\epsilon<0,8>$.

They can be expressed as:

$$\begin{aligned} S_1 &= 2<\hat{n}>+2Re<\hat{a}^2>-4Re(<\hat{a}>)^2\ ; \\ S_2 &= 2<\hat{n}>-2Re<\hat{a}^2>-4Im(<\hat{a}>)^2\ . \end{aligned} \qquad (14)$$

Fig. 3 shows the above squeezing parameters as functions of the nonlinearity χ. The plots are prepared for the same values of α and ξ as for Fig. 1. It is seen that the evolution of $S_{1,2}$ has a similar character as that for the Mandel Q-parameter. We observe collapse and revival-like evolution and the periodic behaviour again. Moreover, for $\chi\sim 2n\pi$, $(n=0,1,...)$ the parameter S_1 corresponding to the quadrature X_1 shows that the state under consideration has squeezing properties. Since all parameters discussed here are constructed using the mean values of the annihilation or creation operators (and their combinations), the periodic character of Q and $S_{1,2}$ will become more evident and clearer as we examine the evolution for mean values of $\hat{a}$ or $\hat{a}^\dagger$ and for their combinations.

Thus, in Fig. 4 the real parts of the mean values of the operators are plotted. Fig. 3a corresponds to $Re<\hat{a}>$, Fig. 3b – $Re<\hat{a}^2>$, Fig. 3c – $Re<\hat{a}^3>$ and Fig. 3d – $Re<\hat{a}^4.>$. Moreover, we assume that $\alpha=4$ and $\xi=0$. This situation corresponds to the Kerr state without displacement. It is seen that for $<\hat{a}>$ oscillations are visible only for $\chi\sim 2\pi n$, $(n=0,1,2...)$. These oscillations exhibit collapse and revival-like character again and are periodic with the period equal to 2π. This fact becomes more evident as we examine Eq.(10) describing the evolution of the mean value of the operator $\hat{a}_{DKerr}$, and not the wave-function $|\Psi_{DKerr}\rangle$. The equation (10) contains periodic functions with the period equal to 2π. This fact leads to the discussed character of the evolution shown in Fig. 4a.

For the cases of $<\hat{a}^k>$, $(k=2,3,4)$ shown in Figs.4b-d we observe additional

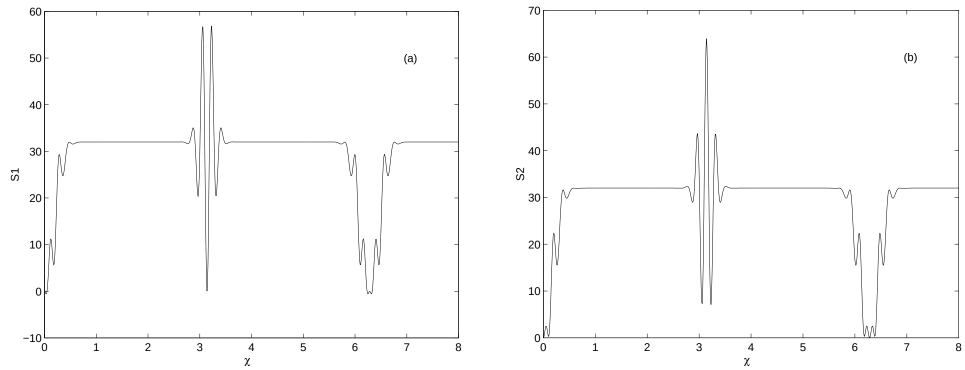

Fig. 3. Squeezing parameters $S_{1,2}$ as function of χ. All parameters are the same as in Fig. 1.

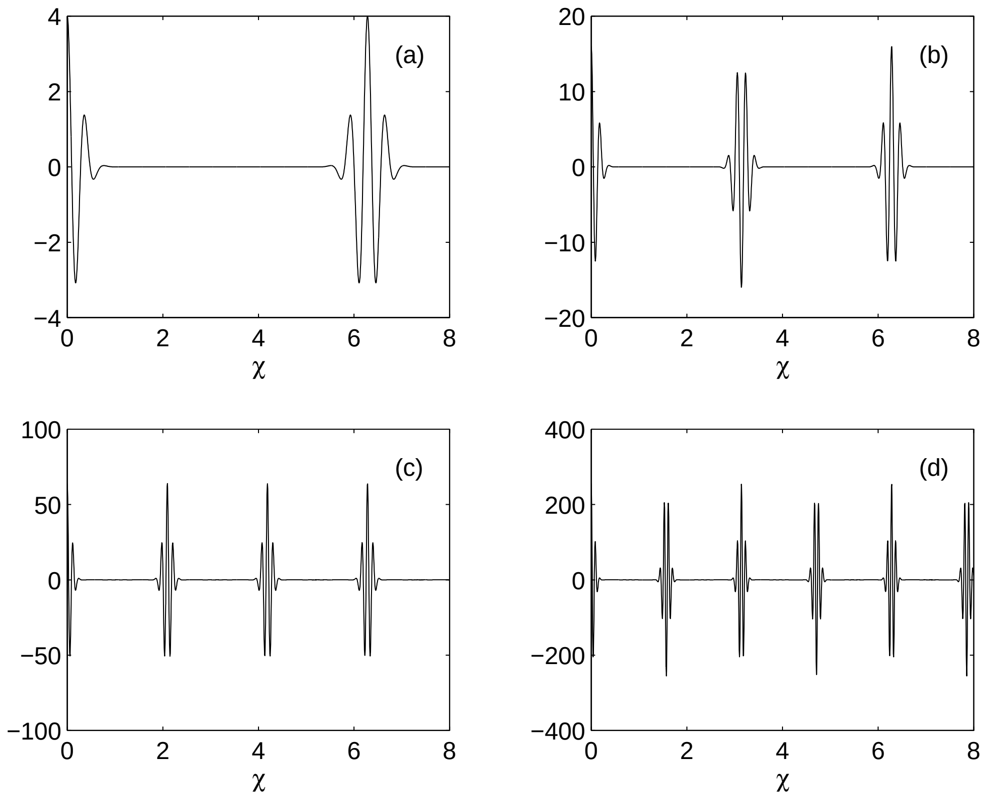

Fig. 4. Real parts of the mean values of the operators: $\hat{a}$ – Fig. 4a, $\hat{a}^2$ – Fig. 4b, $\hat{a}^3$ – Fig. 4c and $\hat{a}^4$ – Fig. 4d. The parameters $\alpha = 4$ and $\xi = 0$.

"revivals" for $\chi = 2j\pi/k$, where $j = 1, ..., k-1$. Moreover, the amplitudes of all the oscillations are identical. In addition, for even powers of the operator $\hat{a}$ we observe phase inversion of subsequent oscillations. Those behaviors can be explained as a result of quantum interference. We calculate mean values of products of the operators, not powers of mean values calculated for single operators.

The situation shown in Fig. 5 corresponds to the same parameters as for Fig. 4, albeit we assume that $\xi = 2$. In consequence, we deal here with DKS. All mean values shown exhibit similar behaviour as those for $\xi = 0$. However, we can observe some influence of the displacement operator. The additional oscillations are less pronounced as ξ becomes significantly greater than zero. Moreover, the amplitudes of satellite oscillations around $\chi = 2n\pi$, $(n = 0, 1, ...)$ are damped and the oscillations around this region of χ become dominant.

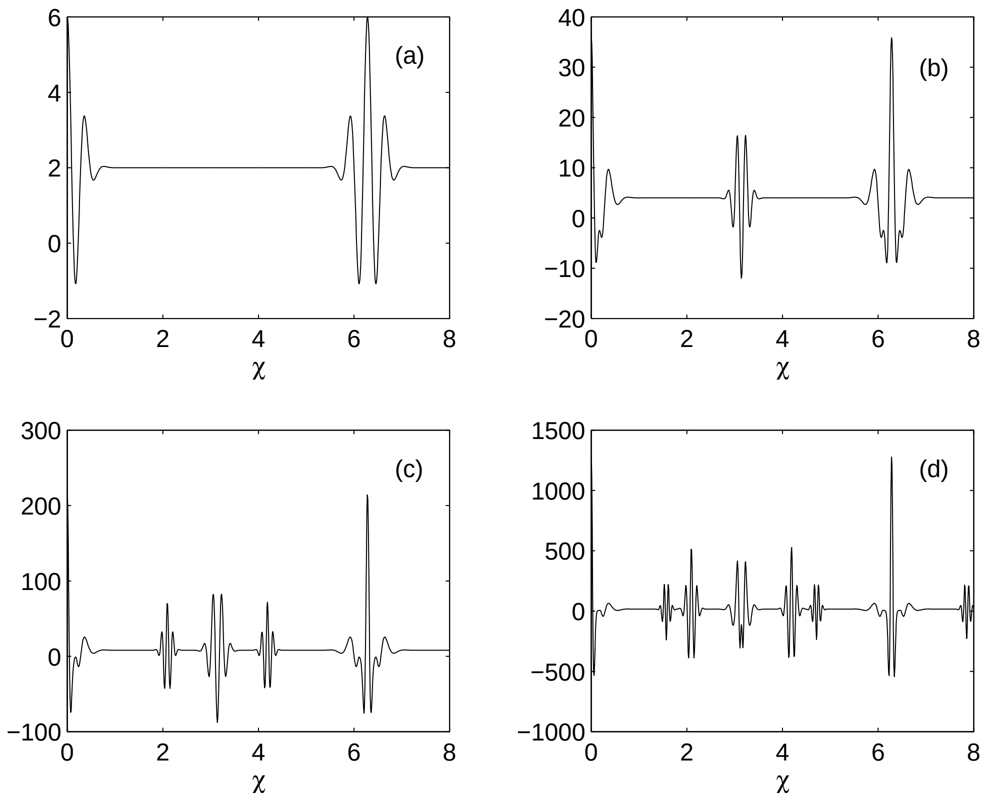

Fig. 5. The same as in Fig. 4 albeit for $\xi = 2$.

It is seen from the above considerations that quantum parameters describing Kerr states and DKS can exhibit behaviour similar to that well known and referred to as collapses and revivals. Nevertheless, one should keep in mind that they are not pure collapses and revivals and this behaviour is typically periodic. We explain this phenomenon as a result of quantum interference. Moreover, additional oscillations can appear for mean values of the higher powers of the operators. For even powers of the operators considered, the subsequent oscillations inverse their phase. The periodic properties of the system can be explained on the basis of the formulas for the evolution of the annihilation operator and for its mean value, contrary to the wave-function formalism where the periodic behaviour was obscured by the rather complicated form of the appropriate equations. Moreover, the expressions for the operator approach are of closed form and contain no summation over n-photon states.

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