acta physica slovaca vol. 48 No. 3, 349 – 360 **June 1998**

STATIONARY STATES IN SATURATED TWO-PHOTON PROCESSES AND GENERATION OF PHASE-AVERAGED MIXTURES OF EVEN AND ODD QUANTUM STATES[1]

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Received 12 May 1998, accepted 22 May 1998

We consider a relaxation of a single mode of the quantized field in a presence of one- and two-photon absorption and emission processes. Exact stationary solutions of the master equation for the diagonal elements of the density matrix in the Fock basis are found in the case of completely saturated two-photon emission. If two-photon processes dominate over single-photon ones, the stationary state is a mixture of phase averaged even and odd coherent states.

1. Introduction

In many cases, the quantum relaxation can be described in the framework of the master equation for the statistical operator $\widehat{\rho}$ [1–3] ($\hbar = 1$)

$$\frac{\partial \widehat{\rho}}{\partial t} + i\left[\widehat{H},\, \widehat{\rho}\right] = \frac{1}{2}\sum_k \left(2\widehat{A}_k\widehat{\rho}\widehat{A}_k^\dagger - \widehat{A}_k^\dagger\widehat{A}_k\widehat{\rho} - \widehat{\rho}\widehat{A}_k^\dagger\widehat{A}_k\right), \tag{1}$$

the $\widehat{A}_k$'s ($k = 1, 2, \cdots$) being some linear operators. If the system under study is an electromagnetic field mode (or an equivalent harmonic oscillator), then Hamiltonian $\widehat{H}$ and each operator $\widehat{A}_k$ can be expressed in terms of the annihilation and creation operators $\widehat{a}$, $\widehat{a}^\dagger$ satisfying the commutation relation $\left[\widehat{a}, \widehat{a}^\dagger\right] = 1$. There exists a specific subfamily of master equations, defined by operators $\widehat{A}_k$ in the form

$$\widehat{A}_k = \left[f_k^{(a)}(\widehat{n})\right]^{1/2}\widehat{a}^k \quad \text{or} \quad \widehat{A}_k = (\widehat{a}^\dagger)^k\left[f_k^{(e)}(\widehat{n})\right]^{1/2}, \tag{2}$$

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$f_k^{(a,e)}(\widehat{n})$ being arbitrary nonnegative functions of the photon number operator $\widehat{n} = \widehat{a}^\dagger \widehat{a}$. If the Hamiltonian is diagonal in the Fock basis, $\widehat{H} = \widehat{H}(\widehat{n})$, then Eqs. (1) and (2) result in a *closed* set of equations for the occupation probabilities $p_n = \langle n|\widehat{\rho}|n\rangle$ $(n = 0, 1, \ldots)$

$$\dot{p}_n = \sum_k \left[n_k^{(+)} f_k^{(a)}(n) p_{n+k} - n_k^{(-)} f_k^{(a)}(n-k) p_n \right]$$
$$- \sum_k \left[n_k^{(+)} f_k^{(e)}(n) p_n - n_k^{(-)} f_k^{(e)}(n-k) p_{n-k} \right], \tag{3}$$

where $n_k^{(+)} \equiv (n+k)!/n!$, $n_k^{(-)} \equiv n!/(n-k)!$. Note that Eq. (3) does not contain off-diagonal matrix elements. And vice versa, the evolution of the off-diagonal elements is completely independent of the evolution of the diagonal ones, since the derivative $\partial\langle m|\widehat{\rho}|n\rangle/\partial t$ is expressed in terms of the elements $\langle m+k|\widehat{\rho}|n+k\rangle$ only (with $k = 0, \pm 1, \pm 2, \ldots$). This means that the stationary solutions to Eqs. (1)-(2) describe completely decoherent states, since all off-diagonal elements relaxate to zero values.

If $f_k^{(a)}$ and $f_k^{(e)}$ are constant positive numbers, then the terms labeled with superscripts $^{(a)}$ or $^{(e)}$ describe the processes of k-photon absorption or emission by some atomic reservoirs [4]. The choice

$$f_k^{(e)}(n) = d_k \left[1 + \gamma_k n_k^{(+)} \right]^{-1}, \quad f_k^{(a)}(n) = const \tag{4}$$

corresponds to a multiphoton generalization of the Scully-Lamb [5] single-mode laser equation, the coefficient γ_k being responsible for the saturation effect. Another equation, implying the presence of emission processes of all orders, described by means of powers of the shift operator $\widehat{u} p_n \equiv p_{n-1} - p_n$, was proposed by Golubev and Sokolov [6]:

$$\dot{p}_n = r \ln(1 + \widehat{u})\, p_n + D_1^{(a)} \left[(n+1) p_{n+1} - n p_n \right]. \tag{5}$$

In this case, coefficients $f_k^{(e)}(n)$ are some rational functions determined by the Taylor expansion of function $\ln(1+u)$. A more general equation, with transcendental (trigonometrical) coefficients $f_k^{(e)}(n)$, was obtained in [7].

It appears that the family of known exact solutions of Eqs. (1) and (3) is not very large. For instance, exact solutions for *arbitrary* functions $f_k^{(a)}(n)$ and $f_k^{(e)}(n)$ were found only in the *stationary* case with $k = 1$ [8]. Exact *time dependent* solutions of Eq. (3) (as well as of equations for the off-diagonal elements) in the case of constant coefficients $f_1^{(a)}$ and $f_1^{(e)}$ were obtained in [9] (see also [10]). Exact time evolution for the one-photon Scully-Lamb equation without absorption ($f_1^{(a)} = 0$) was found in [11].

For $k \geq 2$ (multiphoton processes), exact *time dependent* solutions of Eq. (3) with a *single* nonzero coefficient (either $f_k^{(a)}$ or $f_k^{(e)}$) were found in Refs. [11–21]. In particular, the two-photon emission with $f_2^{(e)} = const$ was considered in [12, 14]. The case of the function $f_2^{(e)}(n)$ in the modified Scully-Lamb form (4) was treated in [11]. The two-photon absorption without emission ($f_2^{(a)} = const$) was studied in detail in [13–17]. The case $f_k^{(a)} = const$ with an arbitrary $k \geq 2$ was investigated in [19, 20], and the case

$f_k^{(e)} = const$ — in [20] (the evolution of the off-diagonal matrix elements in the case of two-photon absorption was studied in [18], and for k-photon absorption — in [20, 21]). Other references can be found, e.g., in [22, 23]. Exact time dependent solutions with *two* nonzero coefficients were obtained in [6,24–26]. In particular, the time dependent absorption problem with constant coefficients $f_1^{(a)}$ and $f_2^{(a)}$ was solved in Ref. [24] (a more detailed analysis was given recently in [26]).

A simplified version of equation (5), with the operator $\ln(1+\widehat{u})$ replaced by the first two terms of the Taylor expansion, $\widehat{u} - \widehat{u}^2/2$, was solved in [6], whereas the solution in a general case was given in [25]. Other exact solutions with two (or more) nonzero coefficients were found in the stationary regime only. For the case of simultaneous k-photon absorption and k-photon emission (the so called systems in detailed balance) this was done in [27] for the coefficients in the form (4), and in [28] for constant $f_k^{(a)}$ and $f_k^{(e)}$ (see also [16] for $k=2$). A scheme of obtaining exact stationary solutions of the two-photon Scully-Lamb equation with single-photon losses ($f_1^{(a)} = const$, $f_2^{(a)} = 0$) was given in [29]. It was generalized to an arbitrary $k \geq 2$ in [30]. A stationary solution the case $f_2^{(e)} = An/(n+2)$, $f_2^{(a)} = Bn(n-1)$, was found in [31] . The case of *three* coefficients, $f_{1,2}^{(a)} = const$, $f_1^{(e)}(n) = const$ or $f_1^{(e)}(n) = A(n+1)^{-1}$, was considered briefly in [32, 33]. A detailed analysis of the problem with three constant coefficients, $f_1^{(a)}$, $f_2^{(a)}$ and $f_1^{(e)}$ (when one has one- and two-photon absorption, but only one-photon emission), was given recently in [34].

The aim of the present article is to find a stationary exact solution to Eq. (3) in the presence of a *two-photon emission*. Although we did not succeed in solving the equation for a *constant* emission coefficient $f_2^{(e)}$, we found that the problem can be solved in the *complete saturation regime* ($\gamma_2 \gg 1$) of the two-photon Scully-Lamb equation (4), when the two-photon emission is described by the function $f_2^{(e)}(n) = D[(n+1)(n+2)]^{-1}$ (with the standard form $f_k^{(a)} = const$ for the absorption terms, $k = 1, 2$). Under this restriction, there exists a 4-parameter family of equations, whose solutions can be expressed in terms of the confluent hypergeometric function or its special cases.

The physical motivation for studying the new model (which is, in turn, a special case of a more general 6-parameter family of equations admitting exact solutions) is explained by the fact that in the case of weak one-photon processes the stationary solutions describe an interesting class of nonclassical states, namely *phase-averaged even and odd states* (PAEOS), which are mixed analogs of the even and odd coherent (pure) states (EOCS)

$$|\alpha_\pm\rangle = N_\pm(|\alpha\rangle \pm |-\alpha\rangle), \quad N_+^2 = \frac{\exp(|\alpha|^2)}{4\cosh(|\alpha|^2)}, \quad N_-^2 = \frac{\exp(|\alpha|^2)}{4\sinh(|\alpha|^2)} \tag{6}$$

($|\alpha\rangle$ means the Glauber coherent state [35]), introduced in [36] and studied, e.g. in [23,37–41] (for generalizations see, e.g. [42–46]). Since EOCS are the simplest examples of the "Schrödinger cat states" (another simple example is the Yurke-Stoler state [47] $|\widetilde{\alpha}\rangle_{YS} = (|\alpha\rangle + i|-\alpha\rangle)/\sqrt{2}$, the principal difference between EOCS and YS-states is that the EOCS have super- (even states) or sub-Poisson (odd states) photon statistics, whereas the statistics of the YS-states is exactly Poissonian), many authors considered different schemes of generating these states in physical processes: see, e.g. [48–52] and an extensive review [53]. It is known, in particular [51], that even and odd coherent states can arise due to the competition between a two-photon absorption and two-photon parametric processes (described by means of a *nondiagonal* Hamiltonian $\widehat{H}(t)$) for a special initial field state. Here we show that one can obtain *phase-averaged* even and odd states using a diagonal Hamiltonian, provided that the (saturated) two-photon emission is admitted.

2. A family of exact solutions

A complete information about the distribution $\{p_n\}$ is contained in the generating function (GF) $F(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$. Its derivatives yield the probabilities p_n and the factorial moments $\mathcal{N}_m \equiv \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)p_n$:

$$p_n = \frac{1}{n!}\left.\frac{\partial^n F}{\partial z^n}\right|_{z=0}, \quad \mathcal{N}_m = \left.\frac{\partial^m F}{\partial z^m}\right|_{z=1}. \tag{7}$$

If the products $n_k^{(\pm)} f_k^{(a,e)}(n)$ are polynomials of n, then one can replace the infinite system of difference equations (3) for the probabilities p_n by a single differential equation for $F(z)$. In the simplest cases, corresponding either to one-photon processes [9, 10], or to a specific form of the emission operator (5) [6, 25], one gets a linear differential equation of the first order. In the most of other known cases, the generating functions satisfy the second order differential equations of the hypergeometric type [13–18,24,26] (an exception is the case considered in [31], where a specific equation of the *fourth* order was solved with the aid of the Laplace method). One can verify that the set of *stationary* ($\dot{p}_n = 0$) equations (3) results in the *second order* equation with *linear* coefficients ($F' \equiv dF/dz$),

$$\begin{aligned} &\left[D_2^{(a)}(1+z) + \left(D_{10}^{(a)} + D_{12}^{(a)}\right) z\right] F'' \\ &+ \left[D_1^{(a)} + 2D_{12}^{(a)} - z\left(D_1^{(e)} + \sum_{j\neq 1} W_{1j}^{(e)}\right)\right] F' \\ &- \left[D_2^{(e)}(1+z) + D_1^{(e)} + D_{11}^{(e)} + \sum_{j\neq 1} j W_{1j}^{(e)}\right] F = 0, \end{aligned} \tag{8}$$

provided that functions $f_k^{(a,e)}$ have the following form:

$$f_2^{(e)}(n) = \frac{D_2^{(e)}}{(n+1)(n+2)}, \qquad f_2^{(a)}(n) = D_2^{(a)}, \tag{9}$$

$$f_1^{(a)}(n) = D_1^{(a)} + D_{10}^{(a)} n + D_{12}^{(a)}(n+2), \tag{10}$$

$$f_1^{(e)}(n) = D_1^{(e)} + \frac{1}{n+1}\left(D_{11}^{(e)} + \sum_{j\neq 1} W_{1j}^{(e)}(n+j)\right), \tag{11}$$

$D_{ij}^{(a,e)}$ and $W_{1j}^{(e)}$ being nonnegative constant coefficients, whereas j can be any integer (excepting the value $j = 1$, which is distinguished for the sake of convenience, because it corresponds to the usual one-photon emission described by the coefficient $D_1^{(e)}$). Since Eq. (8) can be reduced to the Kummer equation [54]

$$xy'' + (c-x)y' - ay = 0, \tag{12}$$

we have a whole family of master equations admitting exact stationary solutions in terms of the confluent hypergeometric function

$$\Phi(a;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(c)_k k!}, \tag{13}$$

where $(a)_n \equiv a(a+1)\cdots(a+n-1)$. This family is determined by 6 independent positive parameters, so it is larger than any one considered until now. Note, however, that we have some freedom only in the choice of terms responsible for the one-photon processes, while the structure of two-photon terms is fixed: the usual two-photon absorption and the completely saturated two-photon emission (corresponding to the limit $\gamma_2 \to \infty$, $d_2/\gamma_2 \to D_2^{(e)}$ in Eq. (4)).

Here we confine ourselves to the special case $D_{10}^{(a)} = D_{12}^{(a)} = W_{1j}^{(e)} = 0$. Then we have 5 independent parameters, $D_{1,2}^{(a,e)}$ and $D_{11}^{(e)}$. Normalizing all the coefficients by the two-photon absorption coefficient, $D_2^{(a)}$, we arrive at the following set of stationary equations for the probabilities and for the generating function:

$$\begin{aligned}&\nu\left\{(n+1)p_{n+1} - np_n - s\left[(n+1)p_n - np_{n-1}\right] - \sigma\left[p_n - p_{n-1}\right]\right\}\\ &+(n+1)(n+2)p_{n+2} - n(n-1)p_n - r^2\left(p_n - p_{n-2}\right) = 0,\end{aligned} \tag{14}$$

$$(1+z)F'' + \nu(1-sz)F' - \left[\nu(s+\sigma) + r^2(1+z)\right]F = 0, \tag{15}$$

where we have introduced the dimensionless coefficients

$$\nu \equiv D_1^{(a)}/D_2^{(a)}, \quad s \equiv D_1^{(e)}/D_1^{(a)}, \quad \sigma \equiv D_{11}^{(e)}/D_1^{(a)}, \quad r^2 \equiv D_2^{(e)}/D_2^{(a)}. \tag{16}$$

A regular solution to Eq. (15) (without a singularity at $z = -1$) satisfying the normalization condition $F(1) = 1$ can be expressed in terms of the confluent hypergeometric function

$$F(z) = e^{h(1-z)} \frac{\Phi(\nu g\,;\, \nu[1+s]\,;\, R[1+z])}{\Phi(\nu g\,;\, \nu[1+s]\,;\, 2R)}, \tag{17}$$

where

$$R = \left[(\nu s)^2 + 4r^2\right]^{1/2}, \quad h = \frac{1}{2}(R - \nu s), \quad g = \frac{1}{R}[s + \sigma + h(1+s)].$$

In particular, if $s=0$, then $R=2r$, $h=r$, and $g=\frac{1}{2}(1+\sigma/r)$. The probabilities and factorial moments can be found with the aid of Eq. (7) and the relation [54]

$$\frac{d^n}{dx^n}\Phi(a;c;x)=\frac{(a)_n}{(c)_n}\Phi(a+n;c+n;x).$$

2.1. Some special cases

If the one-photon processes dominate over two-photon ones, $\nu\to\infty$ (whereas r,s,σ remain finite), then $h\to 0$, $R\approx\nu s$, $\nu g\to 1+\sigma/s$. Replacing $(c)_k$ by c^k in the Kummer series (13) for $c\gg 1$, we obtain the following limit of the generating function (17) as $\nu\to\infty$:

$$F(z)=\left(\frac{1-s}{1-sz}\right)^{1+\sigma/s}. \qquad (18)$$

This is the GF of the *negative binomial distribution*, which was considered in connection with the problems of quantum optics, e.g. in Refs. [55, 56]. For $\sigma=0$ (18) becomes the GF of the thermal (Planck's) distribution, whereas for $s\to 0$ it goes to the GF of the Poisson distribution. Evidently, Eq. (18) is valid only for $s<1$, whereas the general formula (17) holds for all nonnegative parameters s,σ,ν,r. If $s\geq 1$, the asymptotic behaviour of GF at $\nu\gg 1$ is more complicated. For instance, in the particular case $r=\sigma=0$, the distribution $\{p_n\}$ becomes Gaussian when $\nu\gg 1$ and $s\geq 1$ [34].

The Poisson distribution arises also in the limit $s\to\infty$. Then the GF tends to $\exp(z-1)$, i.e. the limit distribution has the mean photon number $\overline{n}=1$, independently on the values of other (finite) parameters, ν,σ,r.

Another simple expression for the GF is obtained in absence of the two-photon absorption, $D_2^{(a)}=0$. Then instead of Eq. (15) we get the first order equation

$$\nu(1-sz)F'-[\nu(s+\sigma)+\rho(1+z)]\,F=0 \qquad (19)$$

($\rho\equiv D_2^{(e)}/D_1^{(a)}$), whose normalized solution reads

$$F(z)=\left(\frac{1-s}{1-sz}\right)^{1+\gamma}\exp\left[\frac{\rho}{s}(1-z)\right],\quad \gamma=\frac{1}{s}\left(\sigma+\rho+\frac{\rho}{s}\right). \qquad (20)$$

3. Phase-averaged even and odd states

Now let us consider the situation, when the two-photon processes dominate over the one-photon counterparts. Suppose first that the one-photon processes are completely absent, i.e. $\nu=\nu s=\nu\sigma=0$. Then the solution of Eq. (15) satisfying the condition $F(1)=1$ reads

$$F(z)=(1-\beta)\frac{\cosh(rz)}{\cosh(r)}+\beta\frac{\sinh(rz)}{\sinh(r)}, \qquad (21)$$

so the occupation probabilities are given by

$$p_{2k}=\frac{(1-\beta)r^{2k}}{(2k)!\cosh(r)},\quad p_{2k+1}=\frac{\beta r^{2k+1}}{(2k+1)!\sinh(r)}. \qquad (22)$$

The distribution (22) is nothing but a combination of the photon distribution functions of the *even and odd coherent states* $|\alpha_+\rangle$ and $|\alpha_-\rangle$ (6) with relative weights $1-\beta$ and β, respectively, provided that $|\alpha|^2$ is identified with the ratio of the two-photon emission and absorption coefficients r. The relative weight of the odd states β is determined by the initial conditions, since there is no correlation between even and odd states: $\beta = \sum_{k=0}^{\infty} p_{2k+1}(0)$.

Using the known Wigner function of the Fock state $|n\rangle\langle n|$ [57, 58]

$$W_n(p,q) = 2(-1)^n \exp\left(-p^2-q^2\right) L_n\left(2p^2+2q^2\right)$$

($L_n(x)$ being the Laguerre polynomial) and the generating function of the Laguerre polynomials [54]

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} L_n(x) = e^z J_0\left(2\sqrt{xz}\right),$$

it is not difficult to write an explicit expression for the Wigner function of the mixed state $\widehat{\rho} = \sum p_n |n\rangle\langle n|$ with the coefficients given by Eq. (22):

$$\begin{aligned} W(p,q;\beta,r) &= \frac{\exp\left(-p^2-q^2\right)}{\sinh(2r)} \left\{ \left[1-(1-2\beta)e^{-2r}\right] I_0\left(\sqrt{8r\left(p^2+q^2\right)}\right) \right. \\ &+ \left. \left[(1-2\beta)e^{2r}-1\right] J_0\left(\sqrt{8r\left(p^2+q^2\right)}\right) \right\}. \end{aligned} \tag{23}$$

Here $J_0(z)$ is the Bessel function and $I_0(z)$ is the modified Bessel function. The Wigner function (23) has zero mean values of the quadratures q and p, and it is very different from the Wigner functions of the pure even/odd coherent states

$$\begin{aligned} W_\pm(p,q;\bar{p},\bar{q}) &= 2N_\pm^2 \left\{ \exp\left[-(q-\bar{q})^2-(p-\bar{p})^2\right] + \exp\left[-(q+\bar{q})^2-(p+\bar{p})^2\right] \right. \\ &\pm \left. 2\exp\left(-q^2-p^2\right)\cos\left[2\left(q\bar{p}-p\bar{q}\right)\right] \right\}, \end{aligned} \tag{24}$$

where $\bar{p}, \bar{q}$ are the mean values of the quadratures in the initial coherent state $|\alpha\rangle$ with $\alpha = \left(\bar{q}+i\bar{p}\right)/\sqrt{2}$. However, assuming $\bar{q} = \sqrt{2r}\cos\varphi$, $\bar{p} = \sqrt{2r}\sin\varphi$ and averaging $W_\pm(p,q;\bar{p},\bar{q})$ over the angle φ according to the formula

$$\widetilde{W}_\pm(p,q;r) \equiv \int_0^{2\pi} \frac{d\varphi}{2\pi} W_\pm(p,q;\sqrt{2r}\cos\varphi,\sqrt{2r}\sin\varphi)$$

we arrive exactly at Eq. (23) with $\beta = 0$ for the even states and $\beta = 1$ for the odd states. For this reason we call the state described by the Wigner function (23) as the *phase-averaged even/odd state* (PAEOS). The phase-averaged *coherent states* were considered in [59] in connection with the problem of a classical limit for the quantum oscillator. The PAEOS are quantum mixtures, since the *purity coefficient*

$$\mu \equiv \mathrm{Tr}(\rho^2) = \frac{1}{2}\left\{ \left(\frac{1-\beta}{\cosh r}\right)^2 \left[I_0(2r)+J_0(2r)\right] + \left(\frac{\beta}{\sinh r}\right)^2 \left[I_0(2r)-J_0(2r)\right] \right\}$$

is less than 1 for $r > 0$. It is a monotonous function of r, whose asymptotics are

$$\mu \approx (1-\beta)^2(1-r^2) + \beta^2(1-r^2/3), \quad r \ll 1, \qquad \mu \approx \left[(1-\beta)^2 + \beta^2\right]/\sqrt{\pi r}, \quad r \gg 1.$$

Nonetheless, PAEOS are nonclassical states, since the Wigner function $W(q,p)$ (23) assumes negative values, as shown in Figs. 1 and 2, where we plot $W(q,p)$ as a function of $x = \sqrt{q^2+p^2}$. If $r > 1$, then the plots corresponding to parameters $1-\beta$ and β have a mirror symmetry with respect to the x-axis for $x < \sqrt{r/2}$, since in this region the contribution of the oscillating function $e^r J_0\left(x\sqrt{8r}\right)$ is dominating (note that $W(0,0;r,\beta) = 2(1-2\beta)$ does not depend on r).

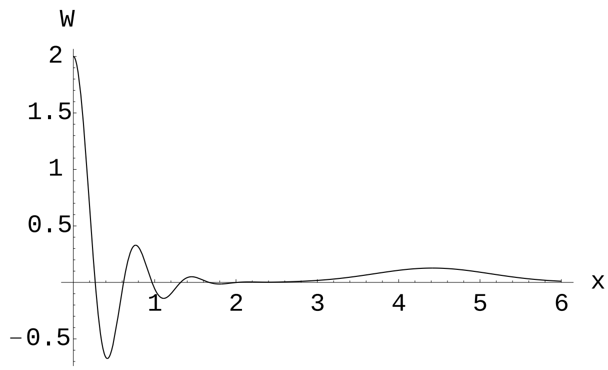

Fig. 1. Wigner function $W(x)$, $x \equiv \sqrt{q^2+p^2}$, of the phase-averaged *even* state ($\beta = 0$) for $r = 10$.

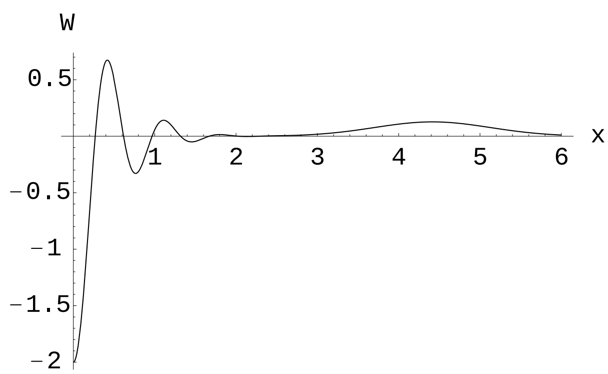

Fig. 2. Wigner function $W(x)$, $x \equiv \sqrt{q^2+p^2}$, of the phase-averaged *odd* state ($\beta = 1$) for $r = 10$.

However, the dependence on β disappears for $x > \sqrt{r/2} > 1$, where the Wigner functions are close to zero in a wide interval (which increases with an increase of r), then exhibit wide and not very high maxima (at $x \approx \sqrt{2r}$), and finally tend to zero exponentially for $x \gg \sqrt{2r}$. In the special case $\beta = 1/2$ the Wigner function (23) is

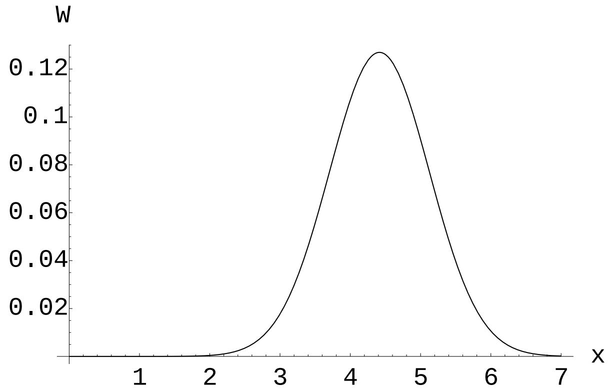

Fig. 3. Wigner function $W(x)$, $x \equiv \sqrt{q^2+p^2}$, of the phase-averaged "maximally mixed state" ($\beta = 0.5$) for $r = 10$.

positive and does not oscillate even for large values of the parameter r, as shown in Fig. 3. Note that the purity coefficient μ also attains its minimum (for a fixed value of r) when β is close to 1/2.

The type of the photon statistics (sub- or super-Poissonian) is determined by Mandel's parameter $\mathcal{Q} \equiv \mathcal{N}_2/\mathcal{N}_1 - \mathcal{N}_1$. In the case of PAEOS this parameter equals

$$\mathcal{Q} = \frac{r}{B}\left(1 - B^2\right), \quad B = (1-\beta)\tanh(r) + \beta\coth(r),$$

so the photon statistics becomes sub-Poissonian for $\beta > \frac{1}{2}\left(1 - e^{-2r}\right)$, i.e. $1 - 2\beta < e^{-2r}$. In particular, in the case $\beta = 1/2$ (Fig. 3) we still have the sub-Poisson statistics.

Till now we assumed that we had no one-photon processes at all. Now let us allow a *small* (but nonzero) coefficient ν (weak one-photon processes). Then we can simplify Eq. (17) with the aid of the relation $\lim_{\nu\to 0}\Phi(a\nu; c\nu; x) = 1 + (a/c)\left(e^x - 1\right)$. In this limit, $R \to 2r$, $h \to r$, $g \to \frac{1}{2}(1+s) + (s+\sigma)/(2r)$, we arrive again at Eq. (21). The essential difference is that now the coefficient β is determined not by the initial conditions, but by the relative strengths of the emission and absorption processes:

$$\beta = \sinh(r)\frac{\sinh(r) + (S/r)\cosh(r)}{\cosh(2r) + (S/r)\sinh(2r)}, \quad S \equiv \frac{s+\sigma}{s+1}, \tag{25}$$

and it is always less than 1/2, since

$$1 - 2\beta = \left[\cosh(2r) + (S/r)\sinh(2r)\right]^{-1} > 0.$$

It is remarkable that parameter ν does not enter the formulas describing the stationary distribution in the limit $\nu \ll 1$. It influences only the relaxation time $t_{rel} \sim \nu^{-1}$, but not the form of the stationary state. If $r \to 0$ (no two-photon emission, two-photon absorption only), then $\beta \to S/(1+2S)$. If we have no one-photon emission ($S \to 0$), then $\beta \to \tanh^2(r)/\left(1 + \tanh^2(r)\right)$. The maximal value $\beta = \frac{1}{2}$ is achieved when $S \to \infty$ or $r \to \infty$.

Mandel's parameter reads now

$$\mathcal{Q} = \frac{r\left[1-(S/r)^2\right]\left[1-\tanh^2(2r)\right]}{[1+(S/r)\tanh(2r)][(S/r)+\tanh(2r)]}. \tag{26}$$

The photon statistics is sub-Poissonian if $r < S$, and super-Poissonian if $r > S$. For a fixed r, function $\mathcal{Q}(S)$ monotonously decreases from $\mathcal{Q}(0) = r\coth(2r)\left[1-\tanh^2(2r)\right]$ to $\mathcal{Q}(\infty) = -r\coth(2r)\left[1-\tanh^2(2r)\right]$. Consequently, $-\frac{1}{2} < \mathcal{Q}(r,S) < \frac{1}{2}$.

4. Conclusion

Let us formulate the main results of the paper. We have found an exactly solvable 6-parameter family of stationary master equations for the diagonal elements of the 1-mode field in a cavity in the presence of competing one- and two-photon emission and absorption processes, and we gave explicit solutions for its 4-parameter subfamily describing the completely saturated two-photon emission regime. We have shown that in the limit case of weak one-photon processes, the field mode goes to the nonclassical stationary state which can be considered as a mixed analog of even and odd pure coherent states. Although we considered an idealized case of a *completely saturated* two-photon emission, the results obtained could help to understand the qualitative features of real (partially saturated) processes.

Acknowledgements This research was supported by FAPESP, project 1995/3843-9. SSM thanks CNPq and FINEP, Brasil, for partial financial support.

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