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PHASE SQUEEZED STATES[1]

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Phase squeezed states of a single mode radiation field have been introduced as eigenstates of a linear combination of lowering and raising operators. The explicit expression in the Fock basis has been obtained and some relevant properties have been illustrated.

1. Introduction

The lack of a Hermitian phase operator also implies the absence of phase eigenstates, namely, physical states of radiation characterized by a definite value of phase. The so-called London phase-states [1]

$$|\varphi\rangle = \sum_{n=0}^{\infty} \exp\{in\varphi\}|n\rangle \ , \tag{1}$$

actually possess infinite energy and can be approached by physical states in different ways. As a consequence, different phase-enhanced states have been suggested and studied in the literature. Among them we mention truncated phase-states [2], intelligent phase-states [ips] and optimized phase-states for a specific, operationally defined, phase detection scheme [3].

Perhaps, the class of states mostly used to describe phase properties of the field is given by field-squeezed states. Squeezed states are eigenstates of a linear combination of annihilation and creation operators, $\mu a + \nu a^\dagger$, with $|\mu|^2 - |\nu|^2 = 1$ and exhibit phase-dependent field fluctuations as well as a number of nonclassical features. In this paper we wish to extend the notion of squeezed state to the phase domain by introducing *phase squeezed* states as eigenstates of a linear combination of lowering and raising operators, $\sigma\hat{\mathcal{E}}_- + \tau\hat{\mathcal{E}}_+$, with $|\sigma|^2 - |\tau|^2 = 1$ and to study their most relevant properties.

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2. Phase Squeezed states

A field-to-phase correspondence can be established by moving from annihilation $a = \sum_k \sqrt{k+1}|k\rangle\langle k+1|$ and creation $a^\dagger$ operators to their formal polar decompositions which are known as lowering $\hat{\mathcal{E}}_- = (a^\dagger a+1)^{-1/2} a = \sum_k |k\rangle\langle k+1|$ and raising $\hat{\mathcal{E}}_+ = \hat{\mathcal{E}}_-^\dagger$ operators. This correspondence cannot be exact, as the canonical commutation relation $[a, a^\dagger] = 1$ is not reproduced by the phase analogue

$$[\hat{\mathcal{E}}_-, \hat{\mathcal{E}}_+] = |0\rangle\langle 0| \ .$$

Thus, it is of interest to analyze the consequences in terms of properties of phase states. The phase analogue of coherent states have been introduced in [4] as the eigenstates of the lowering operator $\hat{\mathcal{E}}_-$: these are the so-called phase coherent states (PCS), whose expression in the Fock basis is given by

$$|\chi\rangle = \sqrt{1-|\chi|^2}\sum_k \chi^k |k\rangle, \qquad \chi \in \mathbf{C}, \quad |\chi| < 1$$

and represent realistic states of radiation, namely, they possess finite energy

$$\langle\chi|a^\dagger a|\chi\rangle = \frac{|\chi|^2}{1-|\chi|^2}$$

and can be synthesized by suitable nonlinear process [5].
The analogue of field-quadratures $\hat{x}_\varphi = 1/2(ae^{-i\varphi} + a^\dagger e^{i\varphi})$ are also known in quantum optics. The phase-quadratures $\hat{\mathcal{E}}_\varphi = 1/2(\hat{\mathcal{E}}_+ e^{i\varphi} + \hat{\mathcal{E}}_- e^{-i\varphi})$ are quantum observables and for the special values $\varphi = 0, \pi/2$ have been received much attention as they coincide with the Susskind-Glogower Sine and Cosine operators, respectively. The eigenstates of phase-quadrature (PQE) are given by

$$|y\rangle_\varphi = \sqrt{\frac{2}{\pi}(1-y^2)}\sum_k U_k(y)e^{ik\varphi}\,|k\rangle, \qquad y \in \mathbf{R}, \quad |y| < 1,$$

$U_k(y)$ being a Chebyshev polynomial of the second kind.

In spite of these facts, no attention has been devoted to the phase analogue of squeezed states. Here we suggest their definition as eigenstates of a linear combination of lowering and raising operators, say, $\hat{K} = \sigma\hat{\mathcal{E}}_- + \tau\hat{\mathcal{E}}_+$, where a convenient parameterization is given by $\sigma = \cosh\rho$ and $\tau = \sinh\rho\, e^{i2\theta}$, and ρ will be called a phase-squeezing parameter. In this way phase squeezed states (PSS) would be states that continuously connect PCS to PQE.

The eigenvalue problem for PSS is given by

$$\hat{K}|\lambda\rangle = \lambda|\lambda\rangle$$

which is equivalent to

$$\begin{cases} \sigma c_1 = \lambda c_0, & k = 0 \\ \sigma c_{k+1} + \tau c_{k-1} = \lambda c_k, & \forall k > 0 \end{cases}, \tag{2}$$

if one uses the following expression for the Fock decomposition for PSS

$$|\lambda\rangle = \mathcal{N}^{-1/2} \sum_k c_k |k\rangle,$$

$\mathcal{N}$ being a normalization constant.
The recursion relation is satisfied for $k > 0$ by Chebyshev polynomials of both kind, whereas the case when $k = 0$ selects Chebyshev polynomials of the second kind. The explicit expression of phase squeezed states in the Fock basis (PSS) are thus given by

$$|\lambda\rangle = \mathcal{N}^{-1/2} \sum_k \left(\frac{\tau}{\sigma}\right)^{k/2} U_k\left(\frac{\lambda}{2\sqrt{\sigma\tau}}\right) |k\rangle \ , \tag{3}$$

with the constraint

$$\lambda \in \mathbf{C}, \qquad |\lambda| < \sigma + |\tau| \equiv e^{\rho} \ . \tag{4}$$

Chebyshev polynomials can be written in terms of Gegenbauer polynomials, $U_n(x) = C_n^1(x)$, so that the normalization constant can be evaluated as follows [6]

$$\mathcal{N} = \sum_{k=0}^{\infty} \left(\frac{|\tau|}{\sigma}\right)^k \left| C_k^1\left(\frac{\lambda}{2\sqrt{\sigma\tau}}\right)\right|^2 = \frac{1-t^2}{A^2 - B^2} \ , \tag{5}$$

where we denote

$$A = 1 + t^2 - 2t|z|^2 \ , \qquad B = 2t|1 - z^2| \ ,$$

and

$$t = \frac{|\tau|}{\sigma} = \tanh\rho \ , \qquad z = \frac{\lambda}{2\sqrt{\sigma\tau}} \ .$$

By writing the PSS amplitude as $\lambda = \sigma\chi + \tau\chi^*$, we can see that

$$|\lambda\rangle \stackrel{\rho\to 0}{\longrightarrow} |\chi \equiv \frac{\lambda}{\sigma}\rangle \qquad \langle\lambda|\, e_-\, |\lambda\rangle = \chi \ , \tag{6}$$

which means that PSS goes to PCS for the phase squeezing parameter going to zero and that such a phase coherent amplitude can be clearly individuated into the PSS amplitude.
For the PSS amplitude being equal to zero we have a phase squeezed vacuum

$$|\rho, \theta\rangle = \frac{1}{\sigma} \sum_{k=0}^{\infty} (-\tanh\rho)^k \ e^{i2k\theta} |2k\rangle \ ,$$

which possesses a residue phase squeezing energy given by $\langle\rho|a^\dagger a|\rho\rangle = 2|\tau|^2$.
The mean photon number of a generic PSS can be expressed in a compact form in terms of the normalization constant $\mathcal{N}$ thought as a function of the variable $t = \tanh\rho$

$$\langle\lambda|a^\dagger a|\lambda\rangle = t\, \partial_t \ln \mathcal{N} \ . \tag{7}$$

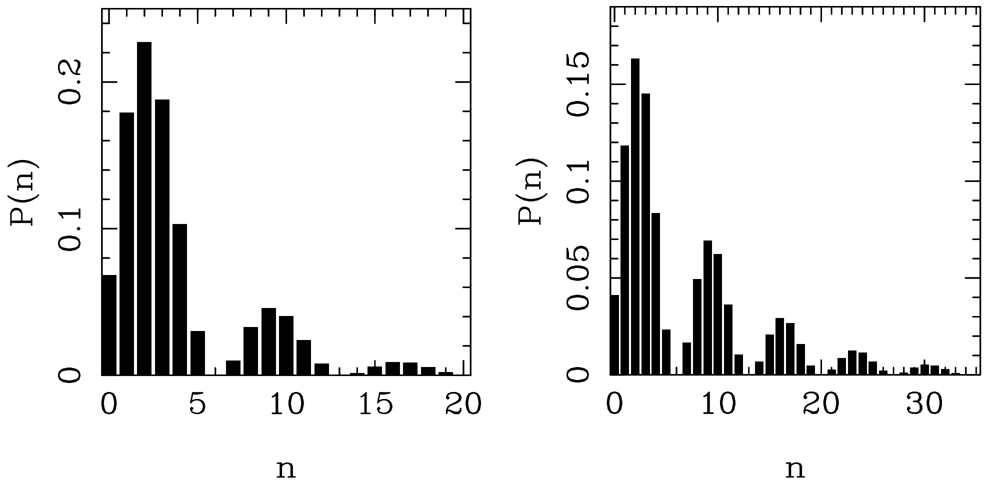

Fig. 1. Photon number distributions of PSS. On the left we show the number distribution of a PSS with $\rho = 1.1$ and $\chi = 0.9$, corresponding to an average photon number given by $\langle a^\dagger a\rangle = 4.0$. On the right, the distribution for $\rho = 1.4$, $\chi = 0.9$, corresponding to $\langle a^\dagger a\rangle = 7.5$. It should be noticed the appearance of oscillations also for small photon number.

In Fig. 1 we report the photon distributions of PSS for different choices of the phase coherent amplitude and squeezing parameter. The distributions exhibit strong oscillations reminding that of field squeezed states [7]. It should be noticed, however, that oscillations in the PSS photon distributions appear also at small energy: actually the distributions plotted in Fig. 1 refer to PSS with $\langle a^\dagger a\rangle = 4.0$ and $\langle a^\dagger a\rangle = 7.5$, respectively.

The peculiar phase properties of PSS, as well as their highly nonclassical features can be visualized by means of the Wigner function. As seen in Fig. 2, the Wigner function of PSS displays peculiar oscillations having a deep negative hollow between pronounced eminences that are caused by interference in the phase space.

It is worth noticing that though PSS are highly nonclassical states the photon number distribution is always super-Poissonian. The Mandel Q-parameter can be evaluated as follows

$$Q \equiv \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle} = -1 + t\,\partial_t \ln\langle\lambda|a^\dagger a|\lambda\rangle\ , \tag{8}$$

where the mean photon number $\langle\lambda|a^\dagger a|\lambda\rangle$ should be thought as a function of the variable t, with expression given in Eq. (7). The Q-parameter appears to be always non-negative thus indicating a super-Poissonian character of the PSS photon statistics. Actually, in contrast to squeezing, the phenomenon of the sub-Poissonian statistics is intensity dependent one. In the case of customary field squeezed state we have sub-Poissonian character when the coherent amplitude exceeds the squeezing parameter. In the case of PSS the constraint (4) doesn't allow us to make such a choice of the parameters and thus to ensure the conditions for the sub-Poissonian statistics.

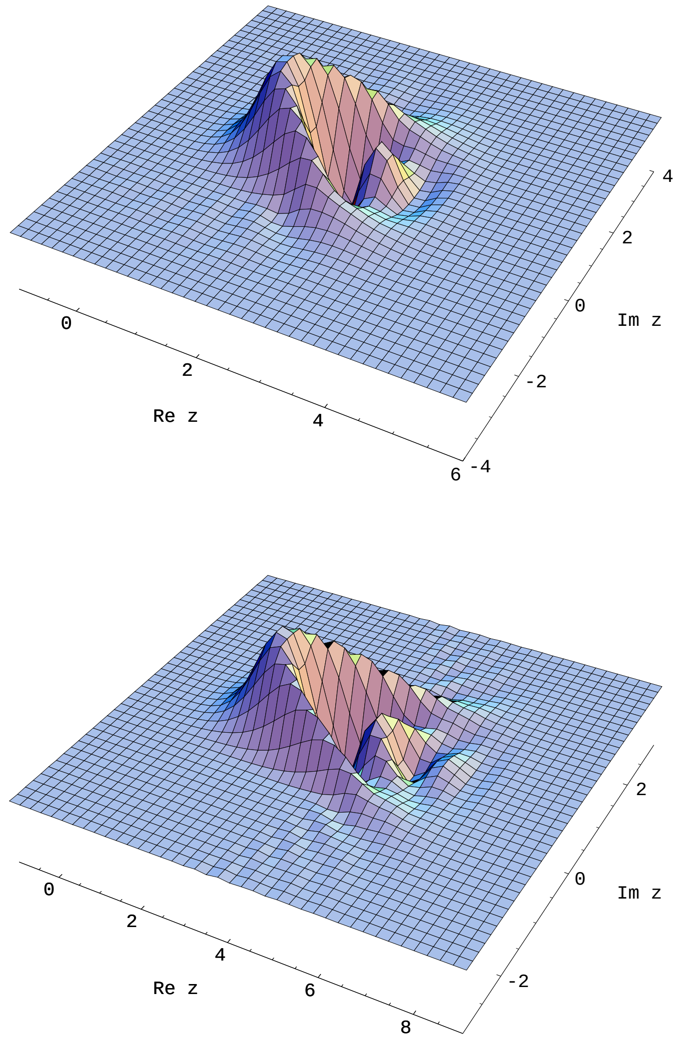

Fig. 2. The Wigner function of PSS. On the top we show the Wigner function of a PSS with $\rho = 1.1$ and $\chi = 0.9$, corresponding to an average photon number given by $\langle a^\dagger a \rangle = 4.0$. The other picture refers to the Wigner function of a PSS for $\rho = 1.5$, $\chi = 0.98$, corresponding to $\langle a^\dagger a \rangle = 10.3$.

3. Conclusions

In the paper, phase squeezed states were defined as eigenstates of a linear combination of phase lowering and raising operators. Their properties were demonstrated by taking as an example evaluation of the mean photon number, the Mandel Q-parameter, photon number distributions and the Wigner function. In particular, strong oscillations in the photon distributions were revealed though such states were shown to possess a super-Poissonian character of the photon statistics.

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