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ON THE CONNECTION BETWEEN THE QUANTUM PHASE SPACE FUNCTIONS OF THE OSCILLATOR AND THE ANGULAR MOMENTUM[1]

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Using Schwinger's model of angular momentum it is possible to introduce general angular momentum phase space distributions by using two independent harmonic oscillators. We establish the connection between general and fixed j phase space functions. This is a reduction based on a relationship between the corresponding coherent states.

1. Introduction

Quasiprobability distributions have already become customary tools of analyzing experimental results in detecting quantum states of systems like an ion oscillating in a harmonic trap, or for a mode oscillator [1]. Recent developments have made it available to work with theoretical constructions of quasiprobabilities for a system of two-level atoms [2]. The best known example for such quasiprobabilities was the P function introduced in [3]. Actually the concept of using quasiprobability functions for systems with a given spin is much older, a Wigner type function has been proposed first by Stratonovich [4] and later on similar constructions have been considered independently by several authors [3, 5, 6, 7, 8]. We use here the construction and notation introduced by Agarwal [5]. Similarly to the case of oscillator quasidistributions [9, 10], the quasiprobability functions for angular momentum states are not unique either.

In the present paper we show a certain connection between these two types of phase space distributions. The method is based on the Schwinger model for angular momentum [11], and the connection between the coherent states of the two systems. As the coherent states are most closely related to the Q function, which is the expectation value of an operator in a coherent state, it is natural that the closest connection that

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can be found is between the Q^h function of a double harmonic oscillator, and the Q^j function of angular momentum in a subspace of fixed angular momentum quantum number. We do not restrict ourselves below to the case of the density operator, for which the phase space representation is a quasiprobability distribution, but we shall allow arbitrary operators, used in the description of the systems in question.

In Section 2 we give a short overview of the angular momentum coherent states based on the Schwinger model. In Section 3 we summarize the method of phase space representations for the oscillator and for angular momentum with fixed value of j. In Section 4 the construction of phase space distributions for general angular momentum is given, and finally in Section 5, we show, how to reduce the general construction of Section 4 to the case of fixed j.

2. Schwinger model and angular momentum coherent states

Let us first recall [12] how one defines angular momentum coherent states based on the Schwinger model [11]. One takes two independent harmonic oscillators: a "+" one and a "−" one, with creation and annihilation operators $a_+^\dagger$, a_+, and $a_-^\dagger$, a_-, respectively. The operators with different subscripts commute among others, while those with identical subscripts satisfy the usual oscillator commutation relations. They act in a Hilbert space spanned by the double number state basis, which are eigenvectors of the number operators $N_+ = a_+^\dagger a_+$, $N = a_-^\dagger a_-$:

$$|n_+, n_-\rangle = \frac{(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}}{\sqrt{n_+!n_-!}}|0,0\rangle. \tag{1}$$

If one defines the following operators:

$$J_+ := a_+^\dagger a_-, \qquad J_- := a_+ a_-^\dagger, \qquad J_3 := \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{1}{2}(N_+ - N_-), \tag{2}$$

it is not difficult to show, that these satisfy the standard angular momentum commutation relations ($\hbar = 1$):

$$[J_+, J_-] = 2J_3, \qquad [J_3, J_\pm,] = \pm J_\pm. \tag{3}$$

This means that the two operator algebras are the same, so the simultaneous eigenvectors of J_3 and J^2 can be labeled by n_+, and n_-. It is simply seen from the definition of J_3 that

$$m = \frac{1}{2}(n_+ - n_-), \qquad j = \frac{1}{2}(n_+ + n_-), \qquad n_+ = j + m, \qquad n_- = j - m. \tag{4}$$

The eigenvalues of J^2 are then, of course, $\frac{1}{2}(n_+ + n_-)(\frac{1}{2}(n_+ + n_-) + 1)$.

So the double oscillator is equivalent to an angular momentum representation, where each $0 \leq j < \infty$ occurs exactly once, with all the possible m-s. The coherent states

$|\,\alpha_+,\alpha_-\rangle$ of the double oscillator can be defined e.g. by their expansion in the number state basis as:

$$|\,\alpha_+,\alpha_-\rangle = e^{-(|\alpha_+|^2+|\alpha_-|^2)/2}\sum_{n_+n_-}\frac{\alpha_+^{n_+},\alpha_-^{n_-}}{\sqrt{n_+!n_-!}}|n_+,n_-\rangle. \tag{5}$$

By rearranging the terms in the doubly infinite sum above, so that first one makes the finite summation over terms where $j=\frac{1}{2}(n_++n_-)$ is fixed, and then over all possible j-s, the identification (4) leads to:

$$|\,\alpha_+,\alpha_-\rangle = e^{-(|\alpha_+|^2+|\alpha_-|^2)/2}\sum_j\frac{1}{\sqrt{(2j)!}}\sum_{m=-j}^{j}\sqrt{\begin{pmatrix}2j\\ j+m\end{pmatrix}}\alpha_+^{j+m},\alpha_-^{j-m}|j,m\rangle =$$

$$= e^{-(|\alpha_+|^2+|\alpha_-|^2)/2}\sum_{j,m}((j+m)!(j-m)!)^{-1}(\alpha_+a_+^\dagger)^{j+m}(\alpha_-a_-^\dagger)^{j-m}|0,0\rangle. \tag{6}$$

These states can be considered as the coherent states for angular momentum. For a fixed j they coincide with the atomic coherent states of [3] apart from a normalization factor. As the definition of certain spin quasiprobabilities, P and Q functions are based on these coherent states, this formula will have a definitive role in what follows.

3. Phase space representations

3.1. Oscillator

In definitions of phase space representations for a single oscillator a fundamental role is played by the continuous operator basis:

$$D(\lambda) = \exp(\lambda a^\dagger - \lambda^* a). \tag{7}$$

labeled by the complex number λ. The characteristic function of an operator in the Hilbert space of a single oscillator is defined as:

$$\chi_A(\lambda) = \mathrm{Tr}(A\,D(\lambda)). \tag{8}$$

The f type phase space distribution of an operator A is then the Fourier transform of the product of the characteristic function $\chi_A(\lambda)$ and of another c-number function $f(\lambda)$:

$$F_A(\alpha;f) = \frac{1}{\pi^2}\int\chi_A(\lambda)f(\lambda)e^{(\alpha\lambda^*-\alpha^*\lambda)}d^2\lambda = \mathrm{Tr}(A\,\Delta(\alpha;f)), \tag{9}$$

where in the second equality we have introduced the f type operator kernel:

$$\Delta(\alpha;f) = \frac{1}{\pi^2}\int f(\lambda)D(\lambda)(e^{(\alpha\lambda^*-\alpha^*\lambda)}d^2\lambda. \tag{10}$$

Specifically the s ordered distribution function [9] is obtained with:

$$f(\lambda) = \exp(s|\lambda|^2/2). \tag{11}$$

As it is well known, for the density operator the special cases of s in (11) correspond to distinguished quasiprobabilities, namely, when $s = 1, -1, 0$ then the function F reduces to the P function, Q function and the Wigner function respectively. For an arbitrary operator A the Wigner function $W_A(\alpha) := F_A(\alpha; f \equiv 1)$, (which is called the Moyal representation[13]), plays a special role, as in that case the following product rule holds:

$$Tr(AB) = \pi \int W_A(\alpha) W_B(\alpha) d^2\alpha. \tag{12}$$

3.2. Angular momentum

A similar procedure has been used earlier for the angular momentum for a fixed j [4, 5, 6, 7] (we use the notation of Ref. [5]). One first chooses an operator basis in the $2j+1$ dimensional Hilbert space, and defines the characteristic function as expansion coefficients in this basis. The most straightforward set of operators is the set of the spherical tensor operators T_{KQ}, which transform among others irreducibly under the action of the rotation operators[14]. Their explicit expression is:

$$T_{KQ} = \sum_{m=-j}^{j} (-1)^{j-m} (2K+1)^{1/2} \begin{pmatrix} j & K & j \\ -m & Q & m-Q \end{pmatrix} |j,m\rangle\langle j, m-Q|, \tag{13}$$

where $\begin{pmatrix} j & K & j \\ -m & Q & m-Q \end{pmatrix}$ is the Wigner $3j$ symbol. Then one introduces the characteristic matrix of any operator A with respect of this operator basis as:

$$A_{KQ} = \mathrm{Tr}(A\, T^{\dagger}_{KQ}), \tag{14}$$

which, according to the orthogonality property $\mathrm{Tr}(T_{K'Q'} T^{\dagger}_{KQ}) = \delta_{KK'} \delta_{QQ'}$ can be inverted yielding the expansion of A in terms of the T_{KQ}-s:

$$A = \sum_{K=0}^{2j} \sum_{Q=-K}^{K} A_{KQ} T_{KQ}. \tag{15}$$

One defines now the following two operator kernels of type Ω and Ω^{-1}:

$$\Delta^j(\theta, \varphi; \Omega) = \sum_{K,Q} T_{KQ} Y^*_{K,Q}(\theta, \varphi) \Omega_{KQ} \ , \tag{16}$$

$$\overline{\Delta}^j(\theta, \varphi; \Omega) = \sum_{K,Q} T_{KQ} Y_{K,Q}(\theta, \varphi) \frac{1}{\Omega_{KQ}} \ . \tag{17}$$

where the Y_{KQ} -s are the spherical harmonics, and the Ω_{KQ}-s are complex numbers depending on the discrete indices K and Q. These operators are self adjoint if $\Omega_{KQ} = \Omega^*_{K,-Q}$. The Ω type quasiprobability distribution for an operator A in the $2j+1$ dimensional Hilbert space is defined with the help of these kernels according to:

$$F^j_A(\theta,\varphi;\Omega) = \frac{1}{\Omega_{00}}\sqrt{\frac{2j+1}{4\pi}}\mathrm{Tr}(A\,\overline{\Delta}^j(\theta,\varphi;\Omega)). \tag{18}$$

It is also useful to introduce the conjugate distribution according to:

$$\overline{F}^j_A(\theta,\varphi;\Omega) = \Omega_{00}\sqrt{\frac{2j+1}{4\pi}}\mathrm{Tr}(A\,\Delta^j(\theta,\varphi;\Omega)). \tag{19}$$

The simplest type of function, which is the analog of the Wigner function of the oscillator is the one in which $\Omega_{KQ} = 1$ for all K and Q. Then the F function is self-conjugate: $F^j_A(\theta,\varphi,1) = \overline{F}^j_A(\theta,\varphi;1) =: W_A(\theta,\varphi)$ (for Wigner) and the product rule can be shown to hold:

$$\mathrm{Tr}(AB) = \int W_A(\theta,\varphi)W_B(\theta,\varphi)\sin\theta d\theta d\varphi. \tag{20}$$

It has become customary to call the expectation value of an angular momentum operator A in an atomic coherent state,

$$|\theta,\varphi\rangle_j = \sum_{m=-j}^{j}\sqrt{\begin{pmatrix} 2j \\ j+m \end{pmatrix}}\sin^{j+m}\frac{\theta}{2}\cos^{j-m}\frac{\theta}{2}e^{-i(j+m)\varphi}\,|j,m\rangle\,, \tag{21}$$

the Q function:

$$Q^j_A(\theta,\varphi) = \frac{2j+1}{4\pi}\,\langle\theta,\varphi|\,A|\theta,\varphi\rangle_j. \tag{22}$$

One can show that the corresponding set of Ω_{KQ}-s is the following:

$$\Omega_{KQ} = \frac{(-1)^{K-Q}\sqrt{4\pi}(2j)!}{\sqrt{(2j-K)!(2j+K+1)!}}\ .$$

.

4. Quasiprobabilities for arbitrary angular momentum

Based on the Schwinger connection between the double oscillator and angular momentum, it is possible to construct phase space distribution functions for general angular momentum operators, as we are going to show below. By the term "general" we mean that the operators, which we want to be represented by the quasiprobability distribution functions, do not need to be restricted to a subspace specified by a particular eigenvalue of the square of the angular momentum.

The construction of these quasiprobability distribution functions is straightforward. In view of the Schwinger model the operators

$$D(\lambda_+,\lambda_-)=D_+(\lambda_+)\ D_-(\lambda_-)\ ,\quad \lambda_+,\lambda_-\in C, \tag{23}$$

form a basis among the general angular momentum operators, therefore any operator A can be expanded as

$$A=\frac{1}{\pi^2}\int d^2\lambda_+\int d^2\lambda_-\frac{1}{f\left(\lambda_+,\lambda_-\right)}\ \chi_A(\lambda_+,\lambda_-)\,D(\lambda_+,\lambda_-), \tag{24}$$

where the f type characteristic function $\chi_A\ (\lambda_+,\lambda_-)$ is the following:

$$\chi_A\left(\lambda_+,\lambda_-\right)=\mathrm{Tr}\left(A\ f\left(\lambda_+,\lambda_-\right)\ D^\dagger(\lambda_+,\lambda_-)\right). \tag{25}$$

Now we introduce the following pair of phase space dependent kernel operators, which also have a functional dependence on the phase space function $f\left(\lambda_+,\lambda_-\right)$:

$$\Delta\left(\alpha_+,\alpha_-;f\right)=\frac{1}{\pi^4}\int d^2\lambda_+\int d^2\lambda_-\ f\left(\lambda_+,\lambda_-\right)D(\lambda_+,\lambda_-)e^{\lambda_+\alpha_+^*-\lambda_+^*\alpha_+}\ e^{\lambda_-\alpha_-^*-\lambda_-^*\alpha_-};$$

$$\begin{aligned}\overline{\Delta}\left(\alpha_+,\alpha_-;f\right)&=\frac{1}{\pi^4}\int d^2\lambda_+\int d^2\lambda_-\frac{1}{f\left(\lambda_+,\lambda_-\right)}D^\dagger(\lambda_+,\lambda_-)e^{\lambda_+^*\alpha_+-\lambda_+\alpha_+^*}\ e^{\lambda_-^*\alpha_--\lambda_-\alpha_-^*}\\&=\Delta^\dagger\left(\alpha_+,\alpha_-;\frac{1}{f^*}\right).\end{aligned}$$

The function f is not specified more closely, except that it should ensure the existence of both definitions. For different f functions we get pairs of different types of quasiprobability distribution functions, characterizing the general angular momentum operator A, by the following definition:

$$F_A\left(\alpha_+,\alpha_-,f\right)=\mathrm{Tr}\left(A\ \Delta\left(\alpha_+,\alpha_-;f\right)\right); \tag{26}$$

$$\overline{F}_A\left(\alpha_+,\alpha_-,f\right)=\mathrm{Tr}\left(A\ \overline{\Delta}\left(\alpha_+,\alpha_-;f\right)\right). \tag{27}$$

It is not difficult to show that

$$\begin{aligned}A&=\pi^2\int d^2\alpha_+\int d^2\alpha_-\overline{F}_A\left(\alpha_+,\alpha_-;f\right)\Delta\left(\alpha_+,\alpha_-;f\right)\\&=\pi^2\int d^2\alpha_+\int d^2\alpha_-\ F_A\left(\alpha_+,\alpha_-;f\right)\overline{\Delta}\left(\alpha_+,\alpha_-;f\right)\end{aligned} \tag{28}$$

The quasiprobability distribution functions F and $\overline{F}$ (with the same f) constitute a pair in the sense that the trace of the product of two operators A and B can be calculated with the help of $F_A\left(\alpha_+,\alpha_-,f\right)$ and $\overline{F}_B\left(\alpha_+,\alpha_-,f\right)$:

$$\mathrm{Tr}(AB)=\pi^2\int d^2\alpha_+\int d^2\alpha_-\ F_A\left(\alpha_+,\alpha_-;f\right)\overline{F}_B\left(\alpha_+,\alpha_-;f\right) \tag{29}$$

as it can be verified by equation (28).

The most well known quasiprobability distribution functions, like the Q, P and Wigner function, can be obtained by using the following particular functions f:

$$
\begin{aligned}
&f_Q(\lambda_+,\lambda_-) = e^{-\left(|\lambda_+|^2+|\lambda_-|^2\right)/2}, &&Q_A^h(\alpha_+,\alpha_-) := F_A\left(\alpha_+,\alpha_-;\ e^{-\left(|\lambda_+|^2+|\lambda_-|^2\right)/2}\right)\\
&f_P(\lambda_+,\lambda_-) = e^{\left(|\lambda_+|^2+|\lambda_-|^2\right)/2}, &&P_A^h(\alpha_+,\alpha_-) := F_A\left(\alpha_+,\alpha_-;\ e^{\left(|\lambda_+|^2+|\lambda_-|^2\right)/2}\right)\\
&f_W\ (\lambda_+,\lambda_-) = 1, &&W_A^h(\alpha_+,\alpha_-) := F_A(\alpha_+,\alpha_-;\ 1)
\end{aligned}
\tag{30}
$$

where the superscript h refers to the double harmonic oscillator. As we can see, the Wigner function is unique in that its pair is itself. If the operator A is a tensor product, i.e. $A = A_+A_-$, where A_+ and A_- act only in the state spaces of the oscillators "+" and "−" respectively, then the corresponding Q, P and Wigner functions factorize.

These definitions are consistent with the well known [9] properties of the P and Q functions:.

$$
Q_A^h(\alpha_+,\alpha_-) = \frac{1}{\pi^2}\ \langle\alpha_+,\alpha_-\,|\ A|\,\alpha_+,\alpha_-\rangle\,; \tag{31}
$$

$$
A = \int d^2\alpha_+ \int d^2\alpha_-\ P_A^h(\alpha_+,\alpha_-)\,|\alpha_+,\alpha_-\rangle\,\langle\alpha_+,\alpha_-|\,, \tag{32}
$$

where $|\alpha_+,\alpha_-\rangle$ is a double coherent state of the double oscillator.

5. Reduction to angular momentum with fixed j

In this section we show that there is a direct connection between the fixed j angular momentum Q function, $Q^j(\theta,\varphi)$, and the arbitrary angular momentum Q function, $Q^h(\alpha_+,\alpha_-)$.

This connection is based on the following important observation concerning fixed j coherent states $|\theta,\varphi\rangle_j$ and arbitrary angular momentum coherent states $|\alpha_+,\alpha_-\rangle$. Expanding both states in the $|j,m\rangle$ basis:

$$
|\theta,\varphi\rangle_j = \sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}\sin^{j+m}\frac{\theta}{2}\cos^{j-m}\frac{\theta}{2}e^{-i(j+m)\varphi}\,|j,m\rangle\,, \tag{33}
$$

$$
|\alpha_+,\alpha_-\rangle = e^{-\left(|\alpha_+|^2+|\alpha_-|^2\right)/2}\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\frac{1}{\sqrt{(2j)!}}\sqrt{\binom{2j}{j+m}}\alpha_+^{j+m}\alpha_-^{j-m}\,|j,m\rangle\,, \tag{34}
$$

the similarity in the expansion coefficients can be noticed. If we project the $|\alpha_+,\alpha_-\rangle$ state onto the fixed j angular momentum subspace with the projector

$$
\Pi_j = \sum_{m=-j}^{j}|j,m\rangle\,\langle j,m|\,, \tag{35}
$$

and choose

$$
\begin{aligned}
\alpha_+ &= e^{-i\varphi}\sin\frac{\theta}{2},\\
\alpha_- &= \cos\frac{\theta}{2},
\end{aligned}
\tag{36}
$$

we get the following important relation:

$$|\theta,\varphi\rangle_j = \sqrt{e\ (2j)!}\ \Pi_j \left| e^{-i\varphi}\sin\frac{\theta}{2}, \cos\frac{\theta}{2} \right\rangle . \tag{37}$$

Now it is easy to infer the connection between $Q^j(\theta,\varphi)$ and $Q^h(\alpha_+,\alpha_-)$, taking into account (31) and (22):

$$Q^j_{A_j}(\theta,\varphi) = \frac{2j+1}{4} e(2j)!\ Q^h_{A_j}\left(e^{-i\varphi}\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right), \tag{38}$$

where

$$A_j = \Pi_j A \Pi_j , \tag{39}$$

is the restriction of the general angular momentum operator A to the fixed j angular momentum subspace. Since equation (38) is valid for arbitrary A_j, the following important relation can be established between the corresponding operator kernels:

$$\Delta^j\left(\theta,\varphi;\Omega^Q\right) = \frac{2j+1}{4} e(2j)!\ \Pi_j \Delta\left(e^{-i\varphi}\sin\frac{\theta}{2}, \cos\frac{\theta}{2}; e^{-\left(|\lambda_+|^2+|\lambda_-|^2\right)/2}\right)\Pi_j, \tag{40}$$

Finally we note that the question of the reduction of an arbitrary f type angular momentum phase space function to a fixed j phase space function arises naturally. This will be the subject of a subsequent publication.

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