HOLSTEIN–PRIMAKOFF $SU(1,1)$ COHERENT STATE IN MICROMASER UNDER INTENSITY–DEPENDENT JAYNES–CUMMINGS INTERACTION[1]

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It is found that the Holstein–Primakoff $SU(1,1)$ coherent state of the cavity field can be generated in a lossless micromaser under the weak Jaynes–Cummings interaction with intensity–dependent coupling of large number of individually injected polarized atoms.

1. Introduction

In last few years the production and detection of nonclassical states in the micromaser cavity have attracted a great deal of attention. There are several schemes that have been proposed to produce number states [1] and the possibility of generating coherent states [2] and so-called tangent and cotangent states [3] has also been predicted by using micromasers in which a quantized field is in a high–Q cavity with injected two-level atoms. Since the work of Kien, Scully and Walther [2] on the generation of coherent states, several generalizations have been suggested by present authors. In particular, by paying attention to multiphoton transition in two–level atoms, there can be produced displacement, squeezing of arbitrary excitations [4,5]. Although foregoing problems have been discussed by utilizing Jaynes–Cummings interaction, one can consider the intensity–dependent Jaynes–Cummings interaction which can provide the Holstein–Primakoff $SU(1,1)$ coherent state [7,8].

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In the present paper we study a resonant, lossless micromaser with injected atoms in a coherent superposition of the upper and lower states. By investigating the weak interaction of large number of individual atoms (in abbreviation, WL–approximation or WLA) and radiation through one–photon resonant transitions in a lossless cavity, we have found a general solution of the recursion relation with arbitrary initial states for the reduced density matrix elements of the cavity field in the first order approximation of the total interaction strength. With the help of further numerical calculations, the behaviour can perfectly be improved. For instance, the numerical calculation provides the strong Holstein–Primakoff $SU(1,1)$ coherent state (HPCS), even it is applicable to the arbitrary fields which are prepared initially in the micromaser cavity. Next section is devoted to the derivation and solution of the recursion relation for the density matrix elements under the WLA, in the same time, in comparison with certain numerical calculations.

2. Quantum evolution of the field state

The Hamiltonian describing the intensity–dependent Jaynes–Cummings model (ID-JCM) in the rotating–wave approximation and in the interaction picture is obtained to be [6]

$$\hat{H} = \hbar g(\hat{R}|a\rangle\langle b| + \hat{R}^{\dagger}|b\rangle\langle a|). \tag{1}$$

Here the operator $|i\rangle\langle j|$ $(i \neq j)$ describes the transition from level j to level i, g is real constant and $\hat{R}$ and $\hat{R}^{\dagger}$ are constructed from the photon annihilation and creation operators $\hat{a}$ and $\hat{a}^{\dagger}$ of the cavity mode,

$$\hat{R} = \hat{a}\sqrt{\hat{n}},\;\; \hat{R}^{\dagger} = \sqrt{\hat{n}}\hat{a}^{\dagger}; \tag{2}$$

where $\hat{n} = \hat{a}^{\dagger}\hat{a}$ is the photon number operator. It is clear that

$$[\hat{R},\hat{n}] = \hat{R},\;\; [\hat{R}^{\dagger},\hat{n}] = -\hat{R}^{\dagger},\;\; [\hat{R},\hat{R}^{\dagger}] = 2\hat{n}+1. \tag{3}$$

One can say that the Hamiltonian (1) effectively describes the intensity–dependent coupling between the atom and the radiation field.

By using the relation $|i\rangle\langle j||k\rangle\langle l| = |i\rangle\langle l|\delta_{kj}$ (here δ_{kj} is the Kronecker symbol), we can easily show that

$$(\hat{H}/\hbar)^{2k} = g^{2k}[(\hat{a}\hat{a}^{\dagger})^{2k}|a\rangle\langle a| + (\hat{a}^{\dagger}\hat{a})^{2k}|b\rangle\langle b|],$$

$$(\hat{H}/\hbar)^{2k+1} = g^{2k+1}[\hat{R}(\hat{a}^{\dagger}\hat{a})^{2k}|a\rangle\langle b| + \hat{R}^{\dagger}(\hat{a}\hat{a}^{\dagger})^{2k}|b\rangle\langle a|],$$

where $k(k \geq 0)$ is integer number. Hence, the time evolution operator $\hat{U}(t)$ of the atom–field system can be expressed in the form [6]

$$\begin{aligned}
\hat{U}(t) &= \exp(-i\hat{H}t/\hbar) \\
&= \cos\left(gt(\hat{n}+1)\right)|a\rangle\langle a| + \cos\left(gt\hat{n}\right)|b\rangle\langle b| \\
&- i\hat{R}\sin(gt\hat{n})\hat{n}^{-1}|a\rangle\langle b| - i\hat{R}^{\dagger}\sin(gt(\hat{n}+1))(\hat{n}+1)^{-1}|b\rangle\langle a|.
\end{aligned} \tag{4}$$

Let the atoms be initially prepared in a coherent superposition of their states,

$$\hat{\rho}_A = \rho_{aa}|a\rangle\langle a| + \rho_{bb}|b\rangle\langle b| + \rho_{ab}|a\rangle\langle b| + \rho_{ba}|b\rangle\langle a| = \sum_{i,j=a,b} \rho_{ij}|i\rangle\langle j|, \tag{5}$$

where

$$\rho_{ii} \geq 0, \quad \rho_{aa} + \rho_{bb} = 1, \quad \rho_{ab} = \rho_{ba}^*, \\ |\rho_{ab}| = |\rho_{ba}| \leq \sqrt{\rho_{aa}\rho_{bb}},$$

and be injected into a lossless microwave cavity at such a rate that at most one atom at a time is present inside the cavity. We assume that the time of the interaction of each atom with the cavity field is much shorter than the lifetime of all the atomic levels. Then the atomic spontaneous decay processes to other levels or other modes can be ignored while an atom is inside of the cavity, which means that the joint evolution of the cavity field and atoms is unitary. For simplicity, we suppose that the injected atoms be prepared in the same superposition of their upper and lower states and have the same velocity and, therefore, interact with the cavity field of the same interaction time, say τ.

Moreover, since there is a free evolution of the field density matrix in the time between the subsequent atoms entering the cavity, i.e., the matrix elements $\rho(n,n')$ acquire an extra phase factor $\exp(i(n-n')\omega\delta t)$, where ω is the cavity resonance frequency and δt is the time between the arrivals of subsequent atoms, we assume here that the time δt is chosen in such a way that $\omega\delta t$ is equal to a multiple of 2π. In this case the extra phase factor due to the free evolution is unity and can be discarded. Otherwise we should take it into account in the overall density matrix evolution. If the atoms were arriving at random times they would meet the cavity field with random phases, and the cavity field phase, which is associated with the non–diagonal elements of the field density matrix, would necessarily become random (only diagonal elements would survive). This assumption is a very serious restriction to the model considered here. It means that atoms should be injected into the cavity in a well controllable way.

Assuming that this is possible, the field density matrix $\hat{\rho}$ evolves according to

$$\hat{\rho}_N = Tr_A[\hat{U}(\tau)\hat{\rho}_A \otimes \hat{\rho}_{N-1}\hat{U}^\dagger(\tau)]. \tag{6}$$

Here $\hat{\rho}_N$ is the density matrix of the field after N atoms have passed through the cavity, Tr_A stands for the trace over the atomic variables. In writing (6) we have assumed that the state of the atom is not measured as it exits the cavity. The number of injected atoms N is considered as a scaled evolution time of the system.

By using (6) together with the expressions (4) and (5), we can easily get for the field density–matrix elements the recursion relation

$$\begin{aligned} \rho_{N+1}(n,n') &= \rho_{aa}[C_{n+1}C_{n'+1}\rho_N(n,n') + S_nS_{n'}\rho_N(n-1,n'-1)] \\ &+ \rho_{bb}[C_nC_{n'}\rho_N(n,n') + S_{n+1}S_{n'+1}\rho_N(n+1,n'+1)] \\ &+ i\rho_{ab}[C_{n+1}S_{n'+1}\rho_N(n,n'+1) - S_nC_{n'}\rho_N(n-1,n')] \\ &- i\rho_{ba}[S_{n+1}C_{n'+1}\rho_N(n+1,n') - C_nS_{n'}\rho_N(n,n'-1)], \end{aligned} \tag{7}$$

where

$$C_n = \cos(g\tau n), \quad S_n = \sin(g\tau n).$$

Given the initial state of the cavity field $\hat{\rho}(0) \equiv \hat{\rho}_{N=0} = \hat{\rho}_0$, the recursion relation (7) allows us to obtain the field density matrix $\rho_N(n,n')$ for any N. It is clear from (7) that the coupling between the off–diagonal matrix elements $\rho_N(n,n\pm1) = \rho^*_N(n\pm1,n)$ occurs only when the atomic coherence is present, $\rho_{ab} = \rho^*_{ba}$ are not zero, if the micromaser starts from the vacuum or from a thermal field and the field phase may not always be random. The recursion relation (7) can be rewritten as

$$\begin{aligned} \rho_N(n,n') &= [\alpha^2 C_{n+1}C_{n'+1} + \beta^2 C_n C_{n'}]\rho_{N-1}(n,n') \\ &+ \beta^2 S_{n+1}S_{n'+1}\rho_{N-1}(n+1,n'+1) \\ &+ \alpha^2 S_n S_{n'}\rho_{N-1}(n-1,n'-1) \\ &+ i\alpha\beta e^{i\phi}C_{n+1}S_{n'+1}\rho_{N-1}(n,n'+1) \\ &+ i\alpha\beta e^{-i\phi}C_n S_{n'}\rho_{N-1}(n,n'-1) \\ &- i\alpha\beta e^{i\phi}S_n C_{n'}\rho_{N-1}(n-1,n') \\ &- i\alpha\beta e^{-i\phi}S_{n+1}C_{n'+1}\rho_{N-1}(n+1,n'), \end{aligned} \tag{8}$$

where $\rho_{aa} = \alpha^2,\ \rho_{bb} = \beta^2,\ \rho_{ab} = \alpha e^{i\phi}\beta,\ \rho_{ba} = \alpha e^{-i\phi}\beta,\quad \alpha^2 + \beta^2 = 1$; with the initial condition $\rho_{N=0}(n,n') = \rho_0(n,n')$ here $\rho_0(n,n')$ stands for the input field state. The method adopted here has firstly been used by Kien *et al.* [2]. So, we see that it is convenient to introduce the definition

$$\rho_N(n,n') = (ie^{-i\phi})^{n'-n}\left(\prod_{l=1}^{n} S_l \prod_{l'=1}^{n'} S_{l'}\right)\alpha^{n+n'}\tilde{\rho}_N(n,n'), \tag{9}$$

where $\prod_{l=1}^{0} \equiv 1$. Substituting this expression into (8), we get the recursion relation

$$\begin{aligned} \tilde{\rho}_N(n,n') &= [\alpha^2 C_{n+1}C_{n'+1} + \beta^2 C_n C_{n'}]\tilde{\rho}_{N-1}(n,n') \\ &+ \alpha^2\beta^2 S^2_{n+1}S^2_{n'+1}\tilde{\rho}_{N-1}(n+1,n'+1) \\ &+ \tilde{\rho}_{N-1}(n-1,n'-1) \\ &- \alpha^2\beta C_{n+1}S^2_{n'+1}\tilde{\rho}_{N-1}(n,n'+1) \\ &+ \beta C_n\tilde{\rho}_{N-1}(n,n'-1) \\ &+ \beta C_{n'}\tilde{\rho}_{N-1}(n-1,n') \\ &- \alpha^2\beta S^2_{n+1}C_{n'+1}\tilde{\rho}_{N-1}(n+1,n'), \end{aligned} \tag{10}$$

with the condition

$$\tilde{\rho}_0(n,n') = (ie^{-i\phi})^{n-n'}\left(\prod_{l=1}^{n} S_l \prod_{l'=1}^{n'} S_{l'}\right)^{-1}\alpha^{-n-n'}\rho_0(n,n'). \tag{11}$$

Let α and $S_n \neq 0$. Note that $\tilde{\rho}(n,n')$ can form a real symmetric matrix and it does not depend on the phase ϕ of the initial atomic state; the dependence of the density matrix of the micromaser field on the phase of the initial atomic state is simply described by the factor $(i\exp(-i\phi))^{n'-n}$ in (9).

Now we consider the case of weak atom–field interaction when

$$g\tau\bar{n} \ll 1, \tag{12}$$

where $\bar{n}$ is average photon number of the cavity field, in the same time, we assume that the variance of the photon–number distribution always to be not too large for all the time. In the first–order approximation the coefficients read

$$S_n \simeq g\tau n, \quad S_n^2 \simeq 0, \quad C_n \simeq 1,$$

and the recursion relation (10) becomes

$$\begin{aligned} \tilde{\rho}_N(n,n') &= \tilde{\rho}_{N-1}(n,n') + \tilde{\rho}_{N-1}(n-1,n'-1) \\ &+ \beta[\tilde{\rho}_{N-1}(n,n'-1) + \tilde{\rho}_{N-1}(n-1,n')], \end{aligned} \tag{13}$$

with the condition (11). The solution of (13) is easily found to be,

$$\begin{aligned} \tilde{\rho}_N(n,n') &= \sum_{k,k'=0}^{N} \sum_{p=0}^{\downarrow k} \frac{N!\beta^{k+k'-2p}}{p!(k-p)!(k'-p)!(N-k-k'+p)!} \\ &\times \tilde{\rho}_0(n-k,n'-k'), \end{aligned} \tag{14}$$

here $\downarrow k$ =min(k,k'). Because this solution is not very convenient to be used for large N, we prefer to use the truncated form

$$\begin{aligned} \tilde{\rho}_N(n,n') &= \sum_{k=0}^{n} \sum_{k'=0}^{n'} \sum_{p=0}^{\downarrow k} \frac{N!\beta^{k+k'-2p}}{p!(k-p)!(k'-p)!(N-k-k'+p)!} \\ &\times \tilde{\rho}_0(n-k,n'-k'). \end{aligned} \tag{15}$$

The relation between the $(p+1)$th and pth terms in the sum on the right-hand side of (15) is

$$\begin{aligned} &\left(\frac{N!\beta^{k+k'-2(p+1)}}{(p+1)!(k-p-1)!(k'-p-1)!(N-k-k'+p+1)!}\right) \\ &\times \left(\frac{N!\beta^{k+k'-2p}}{p!(k-p)!(k'-p)!(N-k-k'+p)!}\right)^{-1} \\ &= \frac{(k-p)(k'-p)}{\beta^2(p+1)(N-k-k'+p+1)} \\ &\le \frac{kk'}{\beta^2(N-k-k')}. \end{aligned} \tag{16}$$

Let $\beta \neq 0$ and $N \gg 1$. As is seen from the relation (16) between the $(p+1)$th term and the pth term in the sum (15), in the region of values of n and n' such that

$$n + n' + \frac{nn'}{\beta^2} \ll N, \tag{17}$$

the term with $p = 0$ in the sum on the right-hand side of (15) dominates. Keeping only the $p = 0$–term and using the approximation $N!/(N-k-k')! \simeq N^{k+k'}$, from this expression one finds

$$\begin{aligned} \rho_N(n,n') \quad \simeq \quad & n!n'! \sum_{k=0}^{n} \sum_{k'=0}^{n'} \frac{(\alpha\beta g\tau N)^{k+k'}}{k!k'!(n-k)!(n'-k')!} \\ & \times (ie^{-i\phi})^{k'-k} \rho_0(n-k, n'-k'), \end{aligned} \tag{18}$$

with the conditions (12) and (17). Particularly, if the micromaser starts from a pure state $|\Psi_0\rangle$, then the cavity field evolves into the pure state $|\Psi\rangle$ as follows,

$$\langle n|\Psi\rangle \simeq \sum_{k=0}^{n} \frac{z^k n!}{k!(n-k)!} \langle n-k|\Psi_0\rangle, \tag{19}$$

where $z = -ie^{i\phi}\alpha\beta g\tau N$. Further, the condition $z \leq 1$ should be valid, prior to requirement of completeness of the pure state, i.e., normalization coefficient of $|\Psi\rangle$ might be close to that of $|\Psi_0\rangle$. We see that our solution does not contain any information about the normalization coefficient.

So far, we have given an analytical treatment of obtaining a weak solution of the recursion relation (8). Note that by using direct numerical solution of (8) the proposed solution permits us to have a good understanding of the further evolution associated with generation of strongly evolved pure states. However, as is seen from (19), the weak displacement z ($|z| \leq 1$) of arbitrary states takes place, fortunately, rather strong displacement has been able to be demonstrated numerically in the WLA. For simplicity, we suppose that the micromaser starts from the HPCS

$$|z_0\rangle = (1 - |z_0|^2)^{1/2} \sum_{n=0}^{\infty} z_0^n |n\rangle,$$

where $|z_0| \leq 1$, then (19) becomes

$$\langle n|\Psi\rangle \simeq (1 - |z_0|^2)^{1/2} (z_0 + z)^n, \tag{20}$$

which defines $|z_0 + z\rangle$ HPCS where $|z| < 1$. With the help of numerical calculations of the field the kth order normalized factorial moments [10] which are $\langle W^k\rangle/(\langle W\rangle^k k!)$ with $\langle W^k\rangle = \langle \hat{a}^{\dagger k}\hat{a}^k\rangle$ and the intensity as functions of z, we can demonstrate the HPCS.

As is seen from Fig. 1, the field normalized factorial moments decrease faster with the increase of its order. As mentioned in papers [2,5], Fig. 2 illustrates cooperative

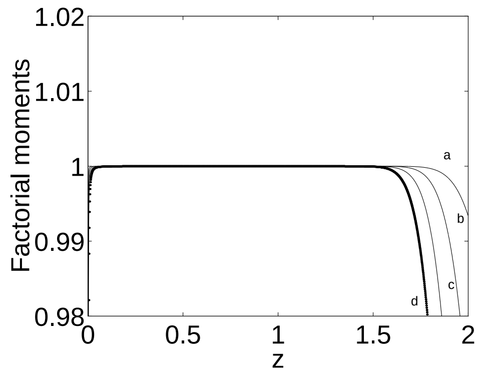

Fig. 1. The field normalized kth order factorial moment $\langle W^k\rangle/(\langle W\rangle^k k!)$ as a function of $z = -ie^{i\phi}\alpha\beta g\tau N, \alpha = \beta = 1/\sqrt{2}, \phi = \pi/2, g\tau = 10^{-4}$ and $N = 40000$ in the case of initial vacuum. The curves a, b, c and d are for $k = 2, 3, 4$ and 5, respectively.

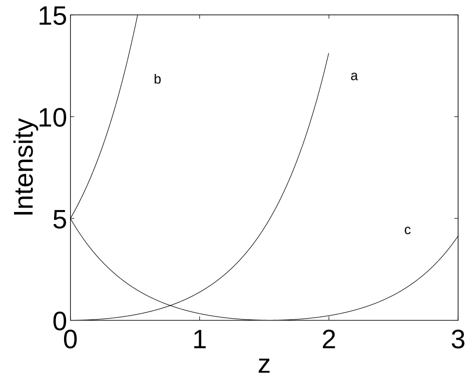

Fig. 2. The field intensity evolution where the parameters are the same as in Fig. 1 (curve a), $N = 15000$ (curve b) and $\phi = 3\phi/2, N = 60000$ (curve c) for initial Holstein–Primakoff $SU(1,1)$ state with mean photon number $\bar{n} = 5$.

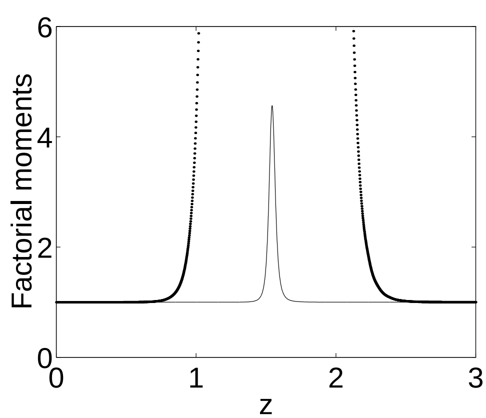

Fig. 3. The second (solid line) and the fifth (dots) order factorial moments with the same parameters as in Fig. 2 for curve c.

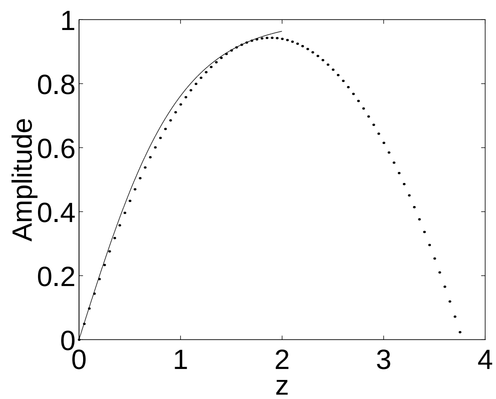

Fig. 4. The field amplitude evolution (solid line) described by $\sqrt{\bar{n}/(\bar{n}+1)}$ and its approximation to the function $f(z) = z - 0.265z^2$ (dots).

behaviour which is provided due to the same initial atomic coherence. The curves a, b and c are associated with superradiance [9] in the presence of HPCS, for initial atomic phase parameters $\pi/2$, $\pi/2$ and $3\pi/2$ and initial field states vacuum, HPCS and HPCS with mean photon number $\bar{n} = 5$, while the atomic numbers $N = 40000$, 15000 and 60000, respectively (all other atomic parameters α and β are always assumed to be

$\alpha = \beta = 1/\sqrt{2}$), with constant initial field phase 2π and interaction time $g\tau = 10^{-4}$. Note that the case of constant initial atomic phase and different initial field phases is the same as previous one. Apart from this, there can be perfect echo, when difference of phases of the amplitudes z_0 and z is out of π (see curve c in Fig. 2 and Fig. 3). Fig. 4 describes deviation between the generated field amplitude which is defined by $\sqrt{\bar{n}/(1+\bar{n})}$ where $\bar{n}$ is the field mean photon number and the analytic function $f(z) = z - 0.265z^2$ associated with the second order approximation in z.

In conclusion, we have discussed that the Holstein–Primakoff $SU(1,1)$ transformation can be provided in the ideal micromaser cavity which is assumed to be realized by intensity–dependent Jaynes–Cummings interaction, if the cavity field starts from arbitrary states and atoms enter the cavity in their coherent superposition of the upper and lower states and the interaction time is short while the number of passed atoms is large enough.

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