QUASIDISTRIBUTIONS FOR FREQUENCY CONVERTER MODEL[1]

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We generalize, for arbitrary initial fields, the Glauber-Mišta theorem of classical-like evolution of the frequency converter. We show, by solving completely the Orlov-Vedenyapin diagonalization problem, that the initially nonclassical fields remain nonclassical during the evolution of the frequency converter. We give a general expression for the two-mode Husimi Q-function and examples of its marginal (single-mode) quasidistributions for initial coherent states, Fock states and two-state superposition of Fock states. We find their graphical representations.

1. Introduction

Parametric frequency converter (PFC) is one of the most fundamental models of quantum optics from both experimental and theoretical points of view. A quantum description of the PFC was given by Louisell [1]. The model has been successfully applied to describe various optical phenomena. In particular, there have been found analogies between the PFC and a beam splitter (see, e.g., [2]), the PFC and a two-level atom driven by a single mode electromagnetic field [3], or the PFC and Raman scattering models [2,4]. By simple generalization of the model various more complicated processes can be described, e.g., coherent or incoherent spontaneous emission from a system of N two-level atoms. There have been great advances in the construction of frequency converters for over 30 years. The PFC devices are based on the coupling of light waves in, e.g., nonlinear dielectric crystals such as $LiNbO_3$.

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2. Model

The parametric frequency converter (PFC) can be modelled by a process of exchanging photons between two optical fields of different frequencies: signal mode at frequency ω_1 and idler mode at frequency ω_2. The interaction Hamiltonian for the PFC is [1]:

$$\widehat{H}_{\rm int} = \hbar\kappa[\widehat{a}_1^\dagger \widehat{a}_2 \exp(-{\rm i}\Delta\omega t) + \widehat{a}_1 \widehat{a}_2^\dagger \exp({\rm i}\Delta\omega t)], \tag{1}$$

where $\Delta\omega = \omega + \omega_2 - \omega_1$; $\widehat{a}_1$ and $\widehat{a}_2$ are the annihilation operators for the signal and idler modes, respectively, and κ is the real coupling constant. Hamiltonian (1) describes a coupling of three optical modes at different frequencies: signal mode at frequency ω_1, idler mode at frequency ω_2, and pump mode at ω. However, in order to derive the Hamiltonian (1) from first principles it is necessary to apply the parametric approximation. This approximation effectively reduces a description of three-mode interaction to a two-mode problem. The pump mode is treated classically since its intensity can be assumed to be much greater than the intensities of the signal and idler modes. The frequency converter (1) is formally equivalent to a beam splitter. For simplicity, we will analyze a resonance case ($\Delta\omega = 0$) only. Solutions of the Heisenberg equations of motion for the signal (1) and idler (2) modes are [1]:

$$\widehat{a}_1(t) = \widehat{a}_{10} \cos\kappa t - {\rm i}\,\widehat{a}_{20} \sin\kappa t, \qquad \widehat{a}_2(t) = \widehat{a}_{20} \cos\kappa t - {\rm i}\widehat{a}_{10} \sin\kappa t, \tag{2}$$

respectively, where $\widehat{a}_{10} = \widehat{a}_1(0)$ and $\widehat{a}_{20} = \widehat{a}_2(0)$ are the annihilation operators at initial moment $t = 0$. The total number of photons is the constant of motion, $\widehat{n}_1(t) + \widehat{n}_2(t) = \widehat{n}_1(0) + \widehat{n}_2(0) = {\rm const}$. The solutions of the classical equations of motion for the PFC are [1]:

$$\Phi_1(\alpha_{10}, \alpha_{20}, t) = \alpha_{10} \cos\kappa t - {\rm i}\alpha_{20} \sin\kappa t, \quad \Phi_2(\alpha_{10}, \alpha_{20}, t) = \alpha_{20} \cos\kappa t - {\rm i}\alpha_{10} \sin\kappa t. \tag{3}$$

3. Diagonalization problem: Sheffer polynomials

Hamiltonian (1) describes a special case of Raman-type models analyzed in the sound paper of Orlov and Vedenyapin [4] on special polynomials in problems of quantum optics. We would like to solve completely their diagonalization problem, $\widehat{H}_{\rm int}|\gamma\rangle = \gamma|\gamma\rangle$, for the PFC model.

Let us assume that the total number of photons in the signal and idler modes is N. Hamiltonian (1) in Fock basis of the idler mode is

$$\widehat{H}_{\rm int} \equiv \widehat{H}_{\rm int}^{(N)} = \begin{bmatrix} 0 & f_0 & 0 & & \\ f_0 & 0 & f_1 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & f_{N-1} \\ & & & f_{N-1} & 0 \end{bmatrix}, \tag{4}$$

where $f_n^{(N)} = \langle N-n-1, n+1|\widehat{a}_1\widehat{a}_2^\dagger|N-n,n\rangle = \sqrt{(n+1)(N-n)}$. The eigenvalues are $\gamma_k \equiv \gamma_k^{(N)} = 2k-N$, where $k=0,...,N$. The eigenvectors

$$|\gamma_k\rangle \equiv |\gamma_k^{(N)}\rangle = \sum_{n=0}^{N} C_n^{(N)}(\gamma_k)|n\rangle, \qquad \text{spanning} \qquad \widehat{H}_{\text{int}}^{(N)} = \sum_{k=0}^{N} \gamma_k|\gamma_k\rangle\langle\gamma_k|, \tag{5}$$

can be determined from the recurrence formula

$$\gamma C_n^{(N)} = f_{n-1}^{(N)} C_{n-1}^{(N)} + f_n^{(N)} C_{n+1}^{(N)} \tag{6}$$

for $n = 0,...,N$. Eq. (6) simplifies to the recurrence formula

$$d_{n+1}^{(N)} = \gamma d_n^{(N)} - n(N-n+1)d_{n-1}^{(N)}, \qquad \text{where} \qquad d_n^{(N)} = C_n^{(N)} n! \sqrt{\binom{N}{n}}, \tag{7}$$

which we recognize as a definition of a special class of the Sheffer orthogonal polynomials [7]. The generating function for the Sheffer polynomials (7) is:

$$\sum_{n=0}^{\infty} d_n^{(N)}(\gamma)t^n = (1+t)^{(N+\gamma)/2}\,(1-t)^{(N-\gamma)/2}\,. \tag{8}$$

Orlov and Vedenyapin [4] found slightly modified form of the generating function (8), however after normalization both functions lead to the same eigenvectors. We complete the analysis of Ref. [4] by finding the explicit solution of the recurrence relation (7) in the form

$$d_n^{(N)}(\gamma) = n! \sum_{j=0}^{n} (-1)^{n-j} \binom{(N+\gamma)/2}{j} \binom{(N-\gamma)/2}{n-j}\,. \tag{9}$$

Normalization constant can be calculated from the Christoffel-Darboux identity for the orthogonal polynomials. By retaining the original coefficients $C_n^{(N)}(\gamma_k) \equiv C_{n,k}^{(N)}$, and putting $\gamma_k = 2k-N$, we arrive at the normalized superposition coefficients (6) in the form

$$C_{n,k}^{(N)} = \sqrt{2^{-N}\binom{N}{k}\binom{N}{n}^{-1}} \sum_{j=0}^{n} (-1)^{n-j} \binom{k}{j} \binom{N-k}{n-j}\,. \tag{10}$$

Knowing the eigenvalues and eigenvectors of the Hamiltonian (4) we readily find the wave function of the frequency converter for initial Fock states $|N-n_0, n_0\rangle$ as

$$|\psi(t)\rangle = \sum_{n=0}^{N} b_n^{(N,n_0)}(t)|N-n,n\rangle, \tag{11}$$

where

$$b_n^{(N,n_0)}(t) = \sum_{k=0}^{N} \exp(-\mathrm{i}\gamma_k\kappa t) C_{n_0}^{(N)}(\gamma_k) C_n^{(N)}(\gamma_k)\ , \tag{12}$$

or, explicitly,

$$b_n^{(N,n_0)}(t) = \sqrt{\binom{N}{n}\binom{N}{n_0}}(\cos\kappa t)^{N-n-n_0}\sum_{j=0}^{n'}\binom{n}{j}\binom{n_0}{j}\binom{N}{j}^{-1}(-\mathrm{i}\sin\kappa t)^{n+n_0-2j}, \tag{13}$$

where $n' = \min(n, n_0)$ or, equivalently, $n' = n$ or $n' = n_0$. We can rewrite the wave function (11) in a more compact form as

$$\begin{aligned}|\psi(t)\rangle &= \frac{\hat{a}_1^\dagger(-t)^{N-n_0}}{\sqrt{N-n_0!}}\frac{\hat{a}_2^\dagger(-t)^{n_0}}{\sqrt{n_0!}}|0,0\rangle \\ &= \frac{(\hat{a}_{10}^\dagger\cos\kappa t\ +\mathrm{i}\,\hat{a}_{20}^\dagger\sin\kappa t)^{N-n_0}}{\sqrt{(N-n_0)!}}\frac{(\hat{a}_{20}^\dagger\cos\kappa t\ +\mathrm{i}\,a_{10}^\dagger\sin\kappa t)^{n_0}}{\sqrt{n_0!}}|0,0\rangle. \end{aligned}\tag{14}$$

4. Quasidistributions

The two-mode Husimi Q-function for arbitrary initial statistics, described by $Q(\alpha_1, \alpha_2, 0) = Q_0(\alpha_{10}, \alpha_{20})$, can be obtained from the explicit form of $b_n^{(N,n_0)}(t)$, given by Eq. (13). With the help of the mathematical identities $|\alpha_1|^2 + |\alpha_2|^2 = |\Phi_1(\alpha_1, \alpha_2, t)|^2 + |\Phi_2(\alpha_1, \alpha_2, t)|^2$ and

$$\Phi_1^{N-n_0}(\alpha_1, \alpha_2, -t) = \sum_{n=0}^{N}\binom{N-n_0}{n-n_0}(\alpha_1\cos\kappa t)^{N-n}(\mathrm{i}\alpha_2\sin\kappa t)^{n-n_0}, \tag{15}$$

for $N \geq n_0$, we show that the following property

$$\langle\psi(t)|\alpha_1, \alpha_2\rangle = \langle N-n_0, n_0|\Phi_1(\alpha_1, \alpha_2, -t), \Phi_2(\alpha_1, \alpha_2, -t)\rangle \tag{16}$$

holds for any initial Fock states, $|\psi(0)\rangle = |N-n_0, n_0\rangle$. By virtue of Eq. (16), we conclude the two-mode Husimi Q-function for arbitrary initial statistics can be expressed in a compact form as

$$\begin{aligned}Q(\alpha_1,\alpha_2,t) &= Q\left\{\Phi_1^{-1}(\alpha_1, \alpha_2, t), \Phi_2^{-1}(\alpha_1, \alpha_2, t), 0\right\} \\ &\equiv Q\left\{\Phi_1(\alpha_1, \alpha_2, -t), \Phi_2(\alpha_1, \alpha_2, -t), 0\right\}, \end{aligned}\tag{17}$$

where

$$\Phi_1^{-1}(\alpha_1, \alpha_2, t) = \alpha_1\cos\kappa t + \mathrm{i}\alpha_2\sin\kappa t, \qquad \Phi_2^{-1}(\alpha_1, \alpha_2, t) = \alpha_2\cos\kappa t + \mathrm{i}\alpha_1\sin\kappa t \tag{18}$$

are the relations inverse to the classical solutions $\Phi_j(\alpha_1, \alpha_2, t)$, given by (3). Obviously, the property holds for any s-parametrized quasidistributions, including the two-mode Glauber-Sudarshan P-function, $P(\alpha_1,\alpha_2,t)$, as was analyzed by Mišta [6] for coherent initial fields.

Eq. (17) has a clear physical interpretation: a two-mode Husimi Q-function for the PFC remains constant along classical trajectories. If the signal and idler fields are initially classical, i.e., if they are described by a regular and non-negative Glauber-Sudarshan P-function, they are classical at any evolution times of the PFC. However, if the fields are initially nonclassical (with singular and/or negative P-function) they remain nonclassical during the evolution. It seems that Glauber [5] and Mišta [6] have proved the property (17) for the model with initial coherent fields only.

In order to find a graphical representation of the PFC evolution is useful to calculate the marginal 3D Q-functions. E.g., the marginal distribution for the signal mode is defined as

$$Q_1(\alpha_1,t) = \int Q(\alpha_1,\alpha_2,t)d^2\alpha_2 = \int Q\left\{\Phi_1^{-1}(\alpha_1,\alpha_2,t),\Phi_2^{-1}(\alpha_1,\alpha_2,t),0\right\}d^2\alpha_2. \quad (19)$$

The idler-mode Husimi function $Q_2(\alpha_2,t)$ is defined analogously. Let us analyze in detail evolution of the PFC for three different initial conditions.

If the PFC is initially in a two-mode coherent state $|\psi(0)\rangle = |\alpha_{10},\alpha_{20}\rangle$, it remains a coherent state at all times [5,6]. From Eq. (17) follows

$$\begin{aligned} Q(\alpha_1,\alpha_2,t) &= \frac{1}{\pi^2}\prod_{j=1,2}\exp(-|\Phi_j^{-1}(\alpha_1,\alpha_2,t)-\alpha_{j0}|^2) \\ &= \frac{1}{\pi^2}\prod_{j=1,2}\exp(-|\alpha_j-\Phi_j(\alpha_{10},\alpha_{20},t)|^2). \end{aligned} \quad (20)$$

The single-mode marginals of $Q(\alpha_1,\alpha_2,t)$ are simply

$$Q_j(\alpha_j,t) = \frac{1}{\pi}\exp(-|\alpha_j-\Phi_j(\alpha_{10},\alpha_{20},t)|^2)\ , \qquad (j=1,2). \quad (21)$$

Let us note that $Q_j(\alpha_j,t)$ differs from $\frac{1}{\pi}\exp(-|\Phi_j^{-1}(\alpha_1,\alpha_2,t)-\alpha_{j0}|^2)$. Eq. (21) shows that the single-mode Husimi functions $Q_j(\alpha_j,t)$ do not change their shape during evolution of initially coherent states (see Fig.1). For better comparison, we present the evolution of the signal and idler modes in the same phase space, i.e., $\alpha_1=\alpha_2$. We have analyzed evolution of initial coherent states, with the same amplitudes $|\alpha_{10}|=|\alpha_{20}|$, but different phases of $\Delta\varphi = \mathrm{Arg}(\alpha_{10})$ and $\mathrm{Arg}(\alpha_{20})=0$. If $\Delta\varphi=0$, then both signal and idler modes evolve along the same circular trajectory given by Eq. (3) with the same phase. However, even by changing slightly $\Delta\varphi$ from, e.g., 0 to 0.1, the differences in the evolution of the modes are well pronounced (see Fig.1a). If $\Delta\varphi=\pi$, then the modes evolve out-of-phase along the same circular trajectory (see Fig.1d). For phases $\Delta\varphi$ different from 0 and π, the trajectories for the signal and idler modes are different (Fig. 1b,c). In particular, for $\Delta\varphi=\pi/2$ (Fig. 1c) and $3\pi/2$, the elliptical trajectories go over into mutually perpendicular linear trajectories.

If the signal mode is initially in a single-photon Fock state and the idler mode in a vacuum state, the wave function is given by $|\psi(t)\rangle = \cos\kappa t|1,0\rangle - \mathrm{i}\sin\kappa t|0,1\rangle$. The two-mode Q-function is

$$Q(\alpha_1,\alpha_2,t) = \frac{1}{\pi^2}\exp(-|\alpha_1|^2-|\alpha_2|^2)|\Phi_1^{-1}(\alpha_1,\alpha_2,t)|^2. \quad (22)$$

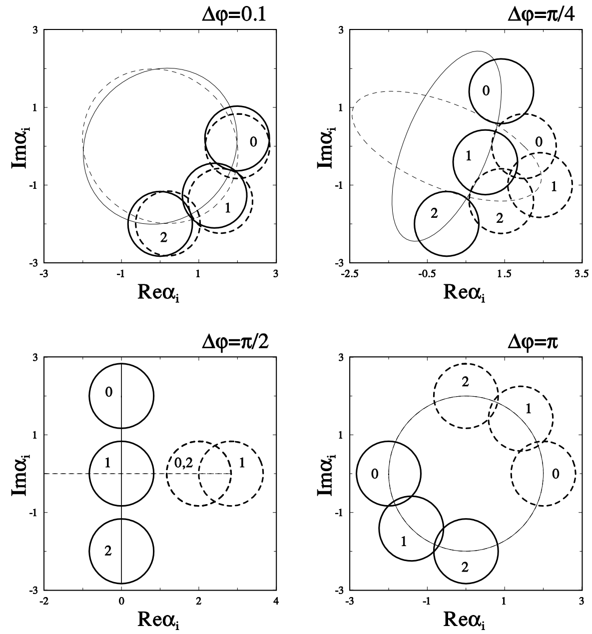

Fig. 1. Contours of the Husimi Q-functions: $Q_1(\mathrm{Re}\alpha_1, \mathrm{Im}\alpha_1, t)$ for the signal mode (thick solid circles) and $Q_2(\mathrm{Re}\alpha_2, \mathrm{Im}\alpha_2, t)$ for the idler mode (thick dashed circles) for initial coherent states $|\psi(0)\rangle = |\alpha_{10}, \alpha_{20}\rangle$ with $\alpha_{10} = 2\exp(\mathrm{i}\Delta\varphi)$, $\alpha_{20} = 2$: (a) $\Delta\varphi = 0.1$, (b) $\Delta\varphi = \frac{\pi}{4}$, (c) $\Delta\varphi = \frac{\pi}{2}$ and (d) $\Delta\varphi = \pi$ at evolution times $\kappa t = 0, 1\cdot\frac{\pi}{4}, 2\cdot\frac{\pi}{4}$. Classical trajectories for the signal mode (thin solid ellipses) and for the idler mode (thin dashed ellipses) are given by Eq.(3).

Its signal-mode marginal is

$$Q_1(\alpha_1, t) = \frac{1}{\pi}\exp(-|\alpha_1|^2)\left(|\alpha_1|^2\cos^2\kappa t + \sin^2\kappa t\right), \tag{23}$$

whereas the idler-mode marginal $Q_2(\alpha_2, t)$ can be obtained from (23) by replacing $\alpha_1 \leftrightarrow \alpha_2$ and $\sin\kappa t \leftrightarrow \cos\kappa t$. The Fig. 2 presents the evolution of the signal-mode Husimi function $Q_1(\alpha_1, t)$ for $\kappa t = 0, \pi, 2\pi, ...$ (Fig. 2a) and for $\kappa t = \pi/2, 3\pi/2, ...$ (Fig. 2b) or equivalently the evolution of the idler-mode Husimi function $Q_2(\alpha_2, t)$ for $\kappa t = \pi/2, 3\pi/2, ...$ (Fig. 2a) and for $\kappa t = 0, \pi, 2\pi, ...$ (Fig. 2b). For $\kappa t = (1+2n)\pi/4$ (n=0,1,...) the Husimi functions coincide, $Q_1(\alpha, t) = Q_2(\alpha, t) = \frac{1}{\pi\sqrt{2}}\exp(-|\alpha|^2)(1 + |\alpha|^2)$ (by assuming the same phase space for both modes). Contrary to the evolution of initially coherent states presented in Fig.1, the Husimi functions $Q_j(\alpha_j, t)$ for initially Fock states are centered at $\alpha_j = 0$ for all evolution times, however they change their shape.

If the signal mode is initially in a superposition of a vacuum and single-photon Fock state, and the idler mode is in a vacuum state, i.e., $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,0\rangle)$, the

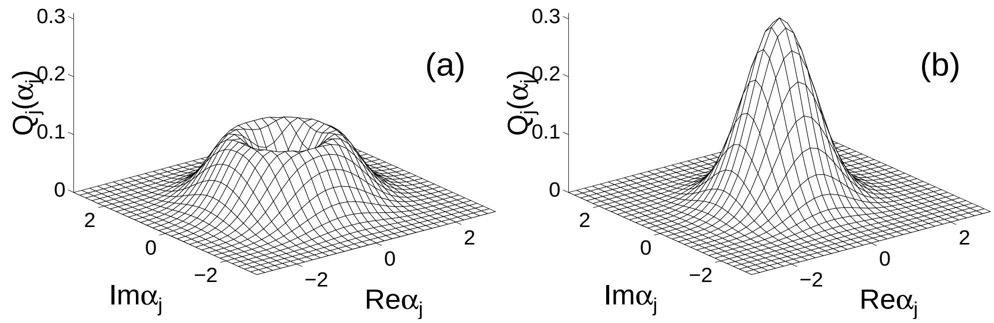

Fig. 2. Husimi Q-functions for initial Fock states $|\psi(0)\rangle = |1,0\rangle$: (a) $Q_1(\mathrm{Re}\alpha_1, \mathrm{Im}\alpha_1, \frac{n\pi}{\kappa})$ or $Q_2(\mathrm{Re}\alpha_2, \mathrm{Im}\alpha_2, (1+2n)\frac{\pi}{2\kappa})$, (b) $Q_1(\mathrm{Re}\alpha_1, \mathrm{Im}\alpha_1, (1+2n)\frac{\pi}{2\kappa})$ or $Q_2(\mathrm{Re}\alpha_2, \mathrm{Im}\alpha_2, \frac{n\pi}{\kappa})$, where $n = 0, 1, \ldots$.

two-mode Q-function is

$$Q(\alpha_1, \alpha_2, t) = \frac{1}{2\pi^2} \exp(-|\alpha_1|^2 - |\alpha_2|^2) \left[1 + |\Phi_1^{-1}(\alpha_1, \alpha_2, t)|^2 + 2\mathrm{Re}\Phi_1^{-1}(\alpha_1, \alpha_2, t)\right]. \tag{24}$$

The signal-mode Husimi function is

$$Q_1(\alpha_1, t) = \frac{1}{2\pi} \exp(-|\alpha_1|^2) \left(|\alpha_1|^2 \cos^2 \kappa t + 2\mathrm{Re}\alpha_1 \cos \kappa t + 1 + \sin^2 \kappa t\right). \tag{25}$$

and the idler-mode Husimi function $Q_2(\alpha_2, t)$ comes from (25) by replacing: $\alpha_1 \to \alpha_2$, $\sin \kappa t \leftrightarrow \cos \kappa t$, and $\mathrm{Re}\alpha_1 \to \mathrm{Im}\alpha_2$. This evolution is $(\frac{2\pi}{\kappa})$-periodical contrary to $(\frac{\pi}{\kappa})$-periodical evolution of initially Fock states $|\psi(0) = |N - n_0, n_0\rangle$ (see, e.g., Fig. 2). The signal-mode Husimi function $Q_1(\alpha_1, t)$ has an apple-shape contour (see Fig. 3 a,b) for $\kappa t = n\pi$ and circular contour for $\kappa t = (1+2n)\pi/2$. The contour of the idler-mode Husimi function $Q_1(\alpha_1, t)$ is initially a circle and it changes into an apple-shape contour at $\kappa t = (1+2n)\pi/2$ but rotated by $\pi/2$ in comparison to $Q_1(\alpha_1, 0)$. Contrary to former cases presented in Figs. 1 and 2, the Husimi functions $Q_j(\alpha_j, t)$ for initial superposition of Fock states (Fig.3) change their shape moving along trajectories.

5. Conclusion

Glauber [5] proved a theorem showing classical behavior of some general class of quantum oscillator systems, including the PFC as a special case. States of the PFC which are initially coherent remain coherent at all times. The evolution of the system is classical in nature for initial coherent states. The similar properties of the PFC were discovered independently by Mišta [6]. He found that the Glauber-Sudarshan P-function is constant along classical trajectories for initial coherent fields.

We have generalized the Glauber-Mišta theorem for arbitrary initial fields. By solving the Orlov-Vedenyapin [4] diagonalization problem completely, we have proved that

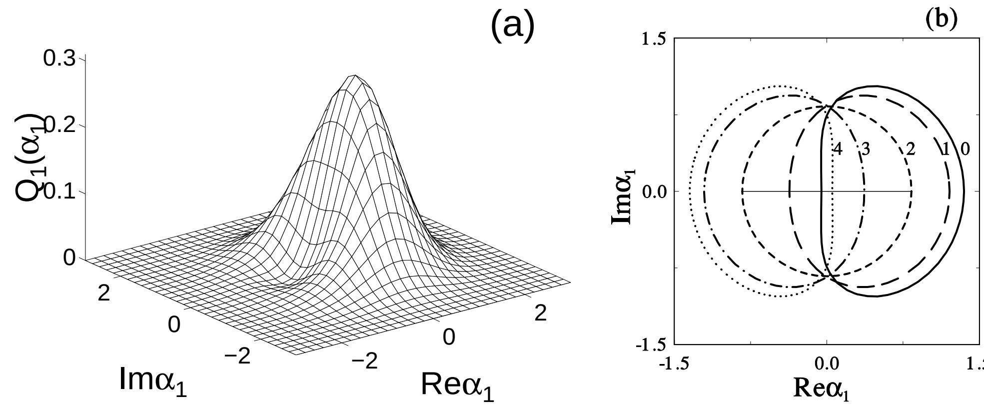

Fig. 3. Husimi Q-functions for initial superposition of Fock states $|\psi(0)\rangle = 2^{-1/2}(|1,0\rangle + |0,0\rangle)$: (a) $Q_1(\mathrm{Re}\alpha_1, \mathrm{Im}\alpha_1, 0)$; (b) contours of $Q_1(\mathrm{Re}\alpha_1, \mathrm{Im}\alpha_1, t)$ at $\kappa t = 0$ [thick solid line - contour corresponding to case (a)], $1 \cdot \frac{\pi}{4}$ (long dash), $2 \cdot \frac{\pi}{4}$ (short dash), $3 \cdot \frac{\pi}{4}$ (dot dash), π (dots). Thin solid line is given by Eq. (3).

the two-mode Husimi Q-functions (or equivalently, the two-mode Glauber-Sudarshan P-functions) for the PFC are constant along classical trajectories for arbitrary superposition of Fock states. It does not imply that the single-mode Husimi Q-functions are constant as well. We have shown that if the initial fields of the PFC are nonclassical, they remain nonclassical during the evolution of the PFC.

We have analyzed in detail three kinds of time evolution of the single-mode Husimi Q-functions for different initial statistics: for single-photon Fock state, for finite and infinite superposition of Fock states initially in the signal mode.

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