RECONSTRUCTION OF PHOTON DISTRIBUTION
WITH POSITIVITY CONSTRAINTS[1]

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An iterative algorithm for reconstructing the photon distribution from the random phase homodyne statistics is discussed. This method, derived from the maximum-likelihood approach, yields a positive definite estimate for the photon distribution with bounded statistical errors even if numerical compensation for the detector imperfection is applied.

A fascinating topic studied extensively in quantum optics over past several years is the measurement of the quantum state of simple physical systems [1]. The central question is how to reconstruct a family of observables characterizing the quantum state from data that can be obtained using a feasible experimental scheme [2]. Most of the initial work on this topic was based on considerations of ideal, noise-free distributions of quantum observables that can be obtained only in the limit of an infinite number of measurements. However, any realistic experiment yields only a finite sample of data. Recognition of the full importance of this fact has led to the development of reconstruction techniques specifically designed to deal with finite ensembles of physical systems [3–6]. The main motivation for these developments is to optimize the amount of information on the quantum state that can be gained from a realistic measurement, and to distinguish clearly between the actual data obtained from an experiment and *a priori* assumptions used in the reconstruction scheme.

One of ideas that have proved to be fruitful in the field of quantum state measurement is the principle of maximum-likelihood estimation. In particular, it has been recently applied to the reconstruction of the photon distribution from the random phase homodyne statistics [7]. The essential advantage of the maximum-likelihood technique is that physical constraints on the quantities to be determined can be consistently built into the reconstruction scheme. This reduces the overall statistical error, and automatically suppresses unphysical artifacts generated by standard linear reconstruction techniques [8], such as negative occupation probabilities of Fock states. However, this is achieved at a significantly higher numerical cost than that required by linear techniques.

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Before we pass to the detailed discussion of the maximum-likelihood method, let us demonstrate its capability to improve the accuracy of the reconstruction. Fig. 1 depicts the final result of processing Monte Carlo simulated homodyne data for a coherent state and a squeezed vacuum state; both states have the average photon number equal to one. We have assumed imperfect photodetection characterized by a quantum efficiency $\eta = 85\%$, which is numerically compensated in the reconstruction scheme. It is seen that the maximum-likelihood estimate, marked with dots, is much closer to the exact distribution than probabilities obtained using the standard linear method of pattern functions. Moreover, purely artificial nonzero values for large photon numbers completely disappear when the maximum-likelihood method is applied.

So, how does the maximum-likelihood method work, and what algorithm provides the bounded, positive definite estimate for the photon distribution? Let us start the discussion from considering the raw data recorded in an experiment. A single experimental run of a random phase homodyne setup yields a certain value of the quadrature observable q. This data, after an appropriate discretization, is recorded as an event in a νth bin. The probability of this event p_ν depends linearly on the photon distribution $\{\varrho_n\}$:

$$p_\nu(\{\varrho_n\}) = \sum_n A_{\nu n} \varrho_n. \tag{1}$$

For a fixed n, the set of coefficients $A_{\nu n}$ describes the homodyne statistics for the nth Fock state.

Repeating the measurement N times yields a frequency histogram $\{k_n\}$ specifying in how many runs the outcome was found in a νth bin. These incomplete, finite data are the source of information on the photon distribution. The question is, how this information can be retrieved. The answer given by the maximum-likelihood estimation method is to pick up the photon distribution for which it was the most likely to obtain the actual result of the performed series of measurements. Mathematically, this is done by the maximization of the log-likelihood function [7]:

$$\mathcal{L}(\{\varrho_n\}) = \sum_\nu k_\nu \ln p_\nu(\{\varrho_n\}) - N \sum_n \varrho_n, \tag{2}$$

where $N = \sum_\nu k_\nu$ is the total number of experimental runs. In the above formula, the method of Lagrange multipliers has been used to satisfy the condition that the sum of all probabilities is equal to one. As the estimate for $\{\varrho_n\}$ is supposed to describe a possible physical situation, the search for the maximum likelihood is *a priori* restricted to the manifold of real probability distributions. Geometrically, this manifold is a simplex defined by the set of inequalities:

$$\varrho_n \geq 0, \qquad n = 0, 1, 2, \ldots$$

$$\sum_n \varrho_n = 1.$$

Thus, the physical constraints on the reconstructed quantities are naturally incorporated in the maximum-likelihood reconstruction scheme.
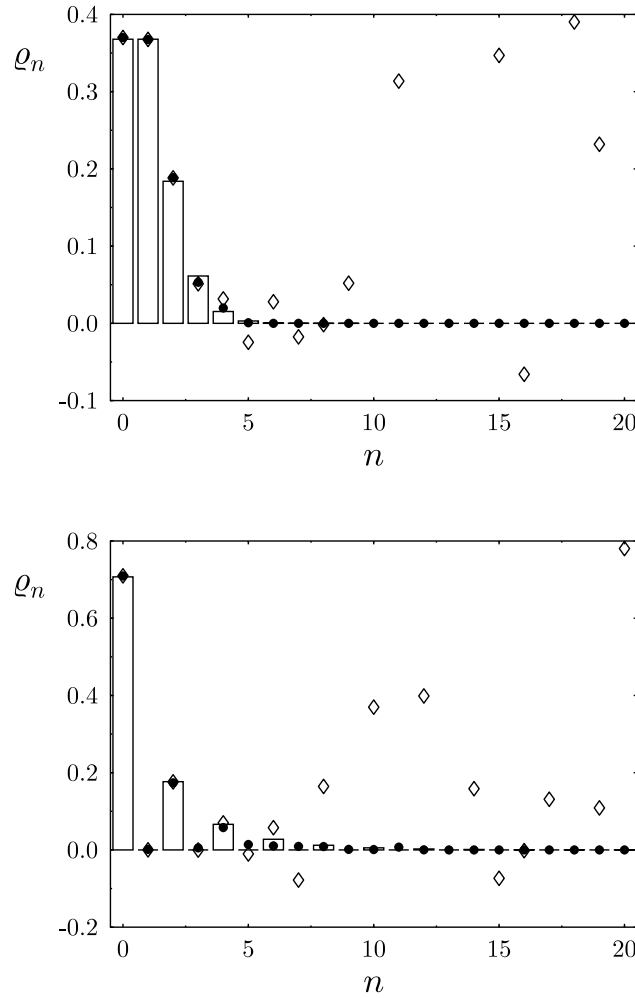
Fig. 1. Reconstruction of the photon distribution from Monte Carlo homodyne data. The bars depict exact photon distributions for a coherent state (top) and a squeezed vacuum state (bottom), both the states with the average photon number equal to one. For each of these states, $N = 10^5$ homodyne events were simulated. The events were divided into 100 bins covering the interval $-5 \leq q \leq 5$. The simulated data were used to reconstruct the photon distributions using the maximum-likelihood method ($\bullet$) and the linear pattern function technique ($\diamondsuit$). In the latter case, some of reconstructed ϱ_n's are beyond the scale of the graphs. Note that for small n, Fock state occupation probabilities reconstructed using both the methods are very close, and the symbols $\bullet$ and $\diamondsuit$ overlap. The maximum-likelihood estimates were obtained from 8000 iterations of the EM algorithm, starting from a uniform distribution for $0 \leq n \leq 20$.

The reconstruction of the photon distribution formulated in the maximum-likelihood approach belongs a very wide class of linear inverse problems with positivity constraints [9]. This class encompasses a variety of problems appearing in as diverse fields as medical imaging and financial markets. As discussed by Vardi and Lee [9], a straightforward method for solving these problems is provided by the so-called expectation-maximization (EM) algorithm. In the following, we will present a heuristic derivation of this algorithm applied the reconstruction of the photon distribution.

Let us consider the necessary condition for the maximum of the function $\mathcal{L}(\{\varrho_n\})$. The partial derivatives of the log-likelihood function are given by:

$$\frac{\partial \mathcal{L}}{\partial \varrho_m} = \sum_\nu k_\nu \frac{A_{\nu m}}{p_\nu(\{\varrho_n\})} - N \tag{3}$$

For each m, the partial derivative $\partial\mathcal{L}/\partial\varrho_m$ must vanish unless the maximum is located on a face of the simplex for which $\varrho_m = 0$. Thus we have that either $\partial\mathcal{L}/\partial\varrho_m = 0$ or $\varrho_m = 0$. These two possibilities can be written jointly as

$$\varrho_m \left(\sum_\nu k_\nu \frac{A_{\nu m}}{p_\nu(\{\varrho_n\})} - N \right) = 0, \qquad m = 0, 1, 2, \ldots. \tag{4}$$

It is convenient to rearrange this condition to the form which defines the maximum-likelihood estimate as a fixed point of a certain nonlinear transformation. Such a form immediately suggests an iterative procedure for finding the fixed point by a multiple application of the derived transformation:

$$\varrho_m^{(i+1)} = \varrho_m^{(i)} \sum_\nu \frac{k_\nu}{N} \frac{A_{\nu m}}{p_\nu(\{\varrho_n^{(i)}\})}, \qquad m = 0, 1, 2, \ldots \tag{5}$$

where upper indices in parentheses denote consecutive approximations for the photon distribution.

In fact, Eq. (5) provides a ready-to-use iterative method for reconstructing the photon distribution, which is a special case of the EM algorithm [9]. Given an approximation of the photon distribution $\{\varrho_n^{(i)}\}$, the next one is simply evaluated according to the right hand side of Eq. (5). When sufficient mathematical conditions are fulfilled, this procedure converges to the maximum-likelihood solution. Thus, the complex problem of constrained multidimensional optimization is effectively solved with the help of a simple iterative algorithm. The maximum-likelihood estimates depicted in Fig. 1 were obtained from 8000 iterations of this algorithm. The initial distributions were assumed to be uniform for photon numbers $n \leq 20$ and equal to zero above this cut-off value.

An essential yet delicate matter in quantum state measurements is the role played by the detection efficiency. The impact of imperfect detection on the maximum-likelihood reconstruction scheme can be understood using a simple intuitive argument. According to Eq. (1), the homodyne statistics is a sum of components generated by the Fock states $|n\rangle$, with the weights given by the appropriate occupation probabilities ϱ_n. The shape of each component is described by the set of coefficients $A_{\nu n}$, with n treated as
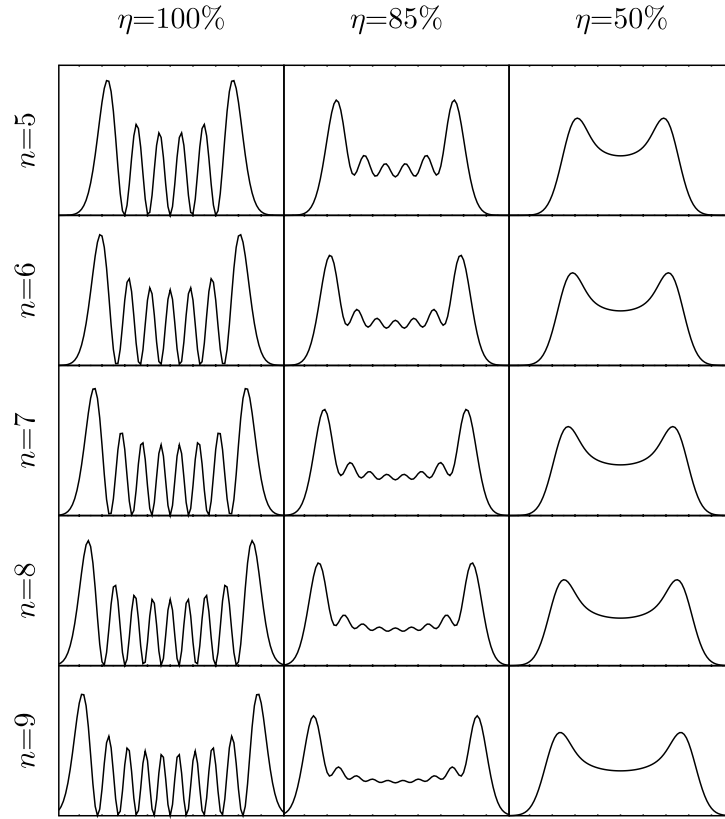
Fig. 2. Components of the random phase homodyne statistics generated by different Fock states, for decreasing efficiency η of the homodyne detector. The homodyne statistics of an arbitrary quantum state is a sum of these components with weights defined by the photon distribution $\{\varrho_n\}$.

a fixed parameter. Fig. 2 depicts several of these components for different values of the detector efficiency η. In the case of unit efficiency, the contribution originating from the Fock state $|n\rangle$ is given by the squared modulus of the nth energy eigenfunction in the position representation, and exhibits characteristic oscillations. The important point is that each of these contributions has a unique location of maxima and minima. Thus, each number state leaves a specific, easily distinguishable trace in the homodyne statistics. Roughly speaking, this is what allows the maximum-likelihood method to resolve clearly the contributions from different number states. When the detection is imperfect, homodyne statistics generated by Fock states become blurred, and oscillations disappear. This makes the shape of the contributions from higher Fock states quite similar. Consequently, it becomes more difficult for the maximum-likelihood method to resolve components generated by different number states. Of course, this intuitive

argument should be supported by quantitative considerations. Mathematically, the effect of imperfect detection is reflected by the shape of the log-likelihood function $\mathcal{L}$. For low detection efficiency, it becomes flatter, and its maximum is not sharply peaked. This increases the statistical uncertainty of the reconstructed photon distribution, and results in a slower convergence of the EM algorithm.

Finally, let us note that the maximum-likelihood algorithm does not make any use of the specific form of the coefficients $A_{\nu n}$ linking the photon distribution with the homodyne statistics. In fact, it can be applied to any phase-insensitive measurement whose statistical outcome depends on the photon distribution via a linear relation of the form assumed in Eq. (1).

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