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# ENTANGLEMENT AND PSEUDOMIXTURES[1]

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In a recent paper Sanpera *et al.* have shown [1], that for the simplest binary composite systems any density matrix can be described in terms of only product vectors. The purpose of this note is to show that possibility of decomposing any state as pseudomixtures does not depend on dimension of the subsystems.

In a composite quantum system one can find states that involve different degrees of correlation between subsystems. From the physical point of view the most important characterization of such states is related to the nature of these correlations. *Separable* states involve correlations which can be explained by classical models [2]. The notion of inseparability or entanglement is related to specific relations existing only in pure quantum systems. Recently a general characterization of separable states was given in the language of vectors from natural cone of the Tomita-Takesaki theory, and quantum origin of this notion was explained [3].

Quantum character of entanglement plays a crucial role in quantum communication [4], cryptography [5], and quantum computation [6]. A state of a composite quantum system is *inseparable* if it cannot by represented as a convex mixture of tensor products of states of its subsystems ($\rho \neq \sum_\omega q_\omega \rho_\omega^{(1)} \otimes \rho_\omega^{(2)}$; otherwise the state is separable). This definition is very hard to handle. Therefore another characterizations are needed. Peres [7] and the Horodeckis [8] have obtained a characterization of separable states for systems described by Hilbert spaces with dimensions $2 \times 2$ or $2 \times 3$. Sanpera *et al.* have shown that for a binary system of dimensions $2 \times 2$ any state can be described as a linear combination of separable states called pseudomixtures.

In this paper we show that any state of any composite quantum system described by Hilbert spaces with finite dimensions can be given in a form of a pseudomixture. In order to show this we use a simple modification of the definition of a separable state [9]:

**Fact 1** *Any density matrix $\rho$ is separable if it is of the form:*

$$\rho = \sum_\alpha p_\alpha P_\alpha \ , \tag{1}$$

*where $p_\alpha > 0$, $\sum_\alpha p_\alpha = 1$ and $P_\alpha$ are projectors on simple-tensor vectors.*

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Let us consider two physical systems described by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. A composite quantum system is described by the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. We need some facts about self-adjoint Hilbert-Schmidt operators ($\rho = \rho^*$ and $\rho \in B_{\mathrm{HS}}(\mathcal{H}_1 \otimes \mathcal{H}_2)$):

**Proposition 1** *For any self-adjoint Hilbert-Schmidt operator $\rho$ and any simple-tensor basis $\{| e_i \otimes f_j\rangle\}_{ij}$ there exists an orthogonal decomposition of $\rho$ into a diagonal component $A$ and an off-diagonal component $H$:*

$$\rho = A + H \quad \text{and} \quad (A, H)_{\mathrm{HS}} = 0, \tag{2}$$

*where $(A,H)_{\mathrm{HS}} = \mathrm{Tr}\, AH$ is a scalar product in $B_{\mathrm{HS}}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.*

**Proof**: Let $\rho = \sum_t p_t P_t$ be the spectral decomposition of $\rho$. Then we have:

$$\begin{aligned} \rho &= \sum_t p_t P_t = \sum_t p_t \mid \sum_{ij} \alpha^t_{ij} e_i \otimes f_j\rangle\langle \sum_{kl} \alpha^t_{kl} e_k \otimes f_l \mid \\ &= \sum_{ijklt} p_t \alpha^t_{ij} \overline{\alpha^t_{kl}} \mid e_i \otimes f_j\rangle\langle e_k \otimes f_l \mid \\ &= \sum_{i,j,t} p_t \mid \alpha^t_{ij}|^2 \mid e_i \otimes f_j\rangle\langle e_i \otimes f_j \mid + \sum_{i\neq k, j\neq l, t} p_t \alpha^t_{ij} \overline{\alpha^t_{kl}} \mid e_i \otimes f_j\rangle\langle e_k \otimes f_l \mid \\ &\equiv A + H\ , \end{aligned} \tag{3}$$

and

$$\begin{aligned} AH &= \sum_{m,n,t} p_t \mid \alpha^t_{mn}|^2 \mid e_m \otimes f_n\rangle\langle e_m \otimes f_n \mid \sum_{i\neq k, j\neq l, s} p_s \alpha^s_{ij} \overline{\alpha^s_{kl}} \mid e_i \otimes f_j\rangle\langle e_k \otimes f_l \mid \\ &= \sum_{m,n,t} p_t \mid \alpha^t_{mn}|^2 \sum_{i\neq k, j\neq l, s} p_s \alpha^s_{ij} \overline{\alpha^s_{kl}} \mid e_m \otimes f_n\rangle\langle e_m \otimes f_n \mid\mid e_i \otimes f_j\rangle\langle e_k \otimes f_l \mid \\ &= \sum_{i\neq k, j\neq l, t, s} p_t p_s \mid \alpha^t_{ij}|^2 \alpha^s_{ij} \overline{\alpha^s_{kl}} \mid e_i \otimes f_j\rangle\langle e_k \otimes f_l \mid . \end{aligned} \tag{4}$$

Therefore $\mathrm{Tr}\, AH = 0$. □

**Proposition 2** *If for any decomposition (2) all diagonal components $A$ are equal to 0, then $\rho$ is equal to 0.*

**Proof**: If we have a decomposition (2) associated with a basis $\{| e_i \otimes f_j\rangle\}_{ij}$, then any other decomposition can be obtained by a unitary transformations $e_i = \sum_k u_{ik} \hat{e}_k$ and $f_j = \sum_l v_{jl} \hat{f}_l$ in the following way:

$$\begin{aligned} P_t &= \mid \sum_{ij} \alpha^t_{ij} e_i \otimes f_j\rangle\langle \sum_{mn} \alpha^t_{mn} e_m \otimes f_n \mid \\ &= \mid \sum_{ij} \alpha^t_{ij} (\sum_{k_1} u_{ik_1} \hat{e}_{k_1}) \otimes (\sum_{l_1} v_{jl_1} \hat{f}_{l_1}) \rangle\langle \sum_{mn} \alpha^t_{mn} (\sum_{k_2} u_{mk_2} \hat{e}_{k_2}) \otimes (\sum_{l_2} v_{nl_2} \hat{f}_{l_2}) \mid \\ &= \mid \sum_{k_1 l_1} (\sum_{ij} \alpha^t_{ij} u_{ik_1} v_{jl_1}) \hat{e}_{k_1} \otimes \hat{f}_{l_1}\rangle\langle \sum_{k_2 l_2} (\sum_{mn} \alpha^t_{mn} u_{mk_2} v_{nl_2}) \hat{e}_{k_2} \otimes \hat{f}_{l_2} \mid \\ &= \sum_{k_1 l_1 k_2 k_1 l_2} \sum_{ijmn} \alpha^t_{ij} \overline{\alpha^t_{mn}} u_{ik_1} v_{jl_1} \overline{u_{mk_2} v_{nl_2}} \mid \hat{e}_{k_1} \otimes \hat{f}_{l_1}\rangle\langle \hat{e}_{k_2} \otimes \hat{f}_{l_2} \mid . \end{aligned} \tag{5}$$

It follows that the operators with the decomposition associated with the basis $\{|\hat{e}_k \otimes \hat{f}_l\rangle\}_{kl}$ (which will be described by $\hat{A}$ and $\hat{H}$) can be given as a function of $\alpha^t_{ij}$ (these coefficients will be referred to as the coefficients of spectral projectors in the basis $\{|e_i \otimes f_j\rangle\}_{ij}$):

$$\hat{A} = \sum_t p_t \sum_{k,l} \sum_{ijmn} \alpha^t_{ij} \overline{\alpha^t_{mn}} u_{ik} v_{jl} \overline{u_{mk} v_{nl}} \mid \hat{e}_k \otimes \hat{f}_l \rangle\langle \hat{e}_k \otimes \hat{f}_l \mid \tag{6}$$

and

$$\hat{H} = \sum_t p_t \sum_{k_1,l_1,k_2 \neq k_1, l_2 \neq l_1} \sum_{ijmn} \alpha^t_{ij} \overline{\alpha^t_{mn}} u_{ik_1} v_{jl_1} \overline{u_{mk_2} v_{nl_2}} \mid \hat{e}_{k_1} \otimes \hat{f}_{l_1} \rangle\langle \hat{e}_{k_2} \otimes \hat{f}_{l_2} \mid . \tag{7}$$

Putting $\beta_{ijmn} = \sum_t p_t \alpha^t_{ij} \overline{\alpha^t_{mn}}$ and using Hermiticity of $\rho$ we obtain:

$$\hat{A} = \sum_{k,l} \sum_{i \geq m, j \geq n} 2\Re(\beta_{ijmn} u_{ik} v_{jl} \overline{u_{mk} v_{nl}}) \mid \hat{e}_k \otimes \hat{f}_l \rangle\langle \hat{e}_k \otimes \hat{f}_l \mid \tag{8}$$

and

$$\hat{H} = \sum_{k_1,l_1,k_2 \neq k_1, l_2 \neq l_1} \sum_{i \geq m, j \geq n} 2\Re(\beta_{ijmn} u_{ik_1} v_{jl_1} \overline{u_{mk_2} v_{nl_2}}) \mid \hat{e}_{k_1} \otimes \hat{f}_{l_1} \rangle\langle \hat{e}_{k_2} \otimes \hat{f}_{l_2} \mid . \tag{9}$$

Assume that all operators $A = 0$ for any decomposition (2). It means that

$$\forall_{u \in U_1, v \in U_2} \forall_{kl} \sum_{i > m, j > n} 2\Re(\beta_{ijmn} u_{ik} v_{jl} \overline{u_{mk} v_{nl}}) = 0, \tag{10}$$

where $U_i$ is the set of all unitary transformations on $\mathcal{H}_i$. We can choose unitary transformations which in the $k_o$-th column has only two nonzero elements $u_{ak_o} = \frac{1}{\sqrt{2}}(\delta_{ai_o} + \delta_{am_o})$ and analogously $v_{bl_o} = \frac{1}{\sqrt{2}}(\delta_{bj_o} + \delta_{bn_o})$. For such $u$ and $v$ we have:

$$\frac{1}{2}\Re(\beta_{i_o j_o m_o n_o}) = 0. \tag{11}$$

Next, choosing $u_{ak_o} = \frac{1}{\sqrt{2}}(i\delta_{ai_o} + \delta_{am_o})$ and $v$ as before, we have

$$\frac{1}{2}\Re(i\beta_{i_o j_o m_o n_o}) = \frac{1}{2}\Im(\beta_{i_o j_o m_o n_o}) = 0. \tag{12}$$

Then $\beta_{i_o j_o m_o n_o} = 0$ for all $i_o \neq m_o$ and $j_o \neq n_o$, so $\rho = 0$. □

**Proposition 3** *If $\rho$ is a density matrix then a norm of the operator $H$ is less than $\frac{1}{2}$ and its trace is equal to 0.*

**Proof**: We can write the operator $H$ in the following form:

$$H = \sum_{i \neq k, j \neq l, t} p_t \alpha^t_{ij} \overline{\alpha^t_{kl}} \mid e_i \otimes f_j \rangle\langle e_k \otimes f_l \mid = \sum_{i > k, j > l, t} p_t G^t_{ijkl}. \tag{13}$$

The operator $G^t_{ijkl}$ has two eigenvalues $\mid \alpha^t_{ij}\alpha^t_{kl} \mid$ and $- \mid \alpha^t_{ij}\alpha^t_{kl} \mid$. Since

$$0 \leq (\mid \alpha^t_{ij} \mid - \mid \alpha^t_{kl} \mid)^2 = \mid \alpha^t_{ij}\mid^2 + \mid \alpha^t_{kl}\mid^2 - 2 \mid \alpha^t_{ij}\alpha^t_{kl} \mid, \tag{14}$$

then

$$\mid \alpha^t_{ij}\alpha^t_{kl} \mid \leq \frac{1}{2}(\mid \alpha^t_{ij}\mid^2 + \mid \alpha^t_{kl}\mid^2). \tag{15}$$

Using this we can estimate the norm of the operator $H$:

$$\begin{aligned} \|H\| &= \| \sum_{i>k,j>l,t} p_t G^t_{ijkl} \| \leq \sum_{i>k,j>l,t} p_t \|G^t_{ijkl}\| = \sum_{i>k,j>l,t} p_t \mid \alpha^t_{ij}\alpha^t_{kl} \mid \\ &\leq \sum_{i>k,j>l,t} p_t \frac{1}{2}(\mid \alpha^t_{ij}\mid^2 + \mid \alpha^t_{kl}\mid^2) \leq \frac{1}{4}\sum_t p_t \sum_{ijkl}(\mid \alpha^t_{ij}\mid^2 + \mid \alpha^t_{kl}\mid^2) = \frac{1}{2}. \end{aligned} \tag{16}$$

The last equality results from $\sum_t p_t = 1$ and $\sum_{ij} \mid \alpha^t_{ij}\mid^2 = 1$. Trace of this operator equals 0:

$$\begin{aligned} 1 &= \operatorname{Tr}\rho = \operatorname{Tr}(\sum_{i,j,t} p_t \mid \alpha^t_{ij}\mid^2 \mid e_i \otimes f_j\rangle\langle e_i \otimes f_j \mid) + \operatorname{Tr} H \\ &= \sum_{i,j,t} p_t \mid \alpha^t_{ij}\mid^2 \operatorname{Tr}(\mid e_i \otimes f_j\rangle\langle e_i \otimes f_j \mid) + \operatorname{Tr} H = \sum_{i,j,t} p_t \mid \alpha^t_{ij}\mid^2 + \operatorname{Tr} H \\ &= \sum_t p_t \sum_{i,j} \mid \alpha^t_{ij}\mid^2 + \operatorname{Tr} H = 1 + \operatorname{Tr} H. \end{aligned} \tag{17}$$

□

Let $\rho$ be any density matrix from $B_{\mathrm{HS}}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For this matrix we can choose such a decomposition (2) that $\operatorname{Tr}\hat{A}^2$ has a maximal value. This maximum exists because we take supremum over $U_1 \otimes U_2$, so over a compact set, of a continuous function of $u$ and $v$:

$$\operatorname{Tr}\hat{A}^2 = \sum_{k,l}(\sum_t p_t \mid \sum_{ij} \alpha^t_{ij} u_{ik} v_{jl}\mid^2)^2, \tag{18}$$

Therefore there exists a basis realizing this maximum, namely $\{\mid e^0_i \otimes f^0_j\rangle\}_{ij}$:

$$\begin{aligned} \rho &= \sum_{i=k,j=l,t_\circ} p_{t_\circ} \mid \beta^{t_\circ}_{ij}\mid^2 \mid e^0_i \otimes f^0_j\rangle\langle e^0_i \otimes f^0_j \mid + \sum_{i\neq k,j\neq l,t_\circ} p_{t_\circ}\beta^{t_\circ}_{ij}\overline{\beta^{t_\circ}_{kl}} \mid e^0_i \otimes f^0_j\rangle\langle e^0_k \otimes f^0_l \mid \\ &\equiv A(1) + H(1) \quad \text{and} \quad \operatorname{Tr} A(1)^2 = \sup_{A\in\mathcal{A}} \operatorname{Tr} A^2, \end{aligned} \tag{19}$$

where $\mathcal{A}$ describe the set of all diagonal components in simple-tensor decomposition, $p_{t_\circ}$ and $\beta^{t_\circ}_{ij}$ are the eigenvalues of $\rho$ and the coefficients of spectral projectors in a basis $\{\mid e^0_i \otimes f^0_j\rangle\}_{ij}$, respectively.

From Proposition 3 we know that $\|H(1)\| < \frac{1}{2}$. The operator $A(1)$ is a linear (in the first step even convex) combination of simple tensors, so we can think about this decomposition as separating out the simple-tensor part. Since $H(1)$ is self-adjoint (as a sum of self-adjoint operators $G^t_{ijkl}$) we can continue the separation procedure. We again choose a decomposition with maximum $\mathrm{Tr}\, A^2$ which is associated with the basis $\{| e^1_i \otimes f^1_j\rangle\}_{ij}$ :

$$H(1) = \sum_{s(1)} o_{s(1)} O_{s(1)} = \sum_{i,j,s(1)} o_{s(1)} \mid \beta^{s(1)}_{ij} |^2 \mid e^1_i \otimes f^1_j\rangle\langle e^1_i \otimes f^1_j \mid$$
$$+ \sum_{i\neq k, j\neq l, s(1)} o_{s(1)} \beta^{s(1)}_{ij} \overline{\beta^{s(1)}_{kl}} \mid e^1_i \otimes f^1_j\rangle\langle e^1_k \otimes f^1_l \mid = A(2) + H(2) \tag{20}$$

After $n$ steps we obtain:

$$\rho = \sum_{i=1}^{n} A(i) + H(n) \tag{21}$$

Since the decomposition is orthogonal, we have

$$\mathrm{Tr}\, H^2(n-1) = \mathrm{Tr}\, A^2(n) + \mathrm{Tr}\, H^2(n) \tag{22}$$

and

$$\sum_{s(n-1)} \mid o_{s(n-1)}|^2 = \sum_{i,j} (\sum_{s(n)} o_{s(n)} \mid \beta^{s(n)}_{ij} |^2)^2 + \sum_{s(n)} \mid o_{s(n)}|^2 > \sum_{s(n)} \mid o_{s(n)}|^2 \tag{23}$$

The equality $\mathrm{Tr}\, H^2(n-1) = \mathrm{Tr}\, H^2(n)$ can be obtained if $A(n) = 0$ and, therefore, by Proposition 2 only for $H(n-1) = 0$. Then a sum of squares of eigenvalues of $H(n)$ is less then a sum of squares of eigenvalues of $H(n-1)$. This means that $\lim_n \|H(n)\| = 0$. In the limit of these procedures we obtain a mixture of simple-tensor projectors $\{| e^u_i \otimes f^u_j\rangle\langle e^u_i \otimes f^u_j \mid\}_{iju}$, not necessarily orthogonal to one another, with real coefficients. If we collect positive and negative coefficients separately we obtain a pseudomixture:

$$\rho = a\rho^{sep(+)} - b\rho^{sep(-)}, \tag{24}$$

where $a, b \geq 0$, $a = 1 + b$, and $\rho^{sep(+)}$ and $\rho^{sep(-)}$ are separable states. For the simplest binary composite systems, it was shown [1] that by using geometrical properties typical of this dimension it is possible to determine cardinality of $\rho^{sep(+)}$ and $\rho^{sep(-)}$ i.e. the smallest $\alpha$ in the decomposition (1). Our procedure does not give such an information. The decomposition (24), similarly to other decompositions of this kind, is not unique. The most interesting pseudomixture is the one for which $b$ is minimal. Our procedure does not determine such a pseudomixture.

Since in our procedure we use only the properties of the Hilbert-Schmidt scalar product, it seems that it can be transferred also to composite systems described by infite-dimensional Hilbert spaces. The language of vectors from natural cone of the Tomita-Takesaki theory is the most natural one for such a problem. The infinite-dimensional case will be discussed elsewhere [10].

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