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# REMARKS ON SEARCH ALGORITHMS AND NONLINEARITY[1]

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Nonlinear data search algorithms proposed recently by Abrams and Lloyd [3] are fast but make an explicit use of an arbitrarily fast unphysical transfer of information within a quantum computer. It is shown that the algorithms can be described also in a fully local formalism.

## 1. Introduction

Nonlinearly evolving quantum computer is a device which, in spite of its purely hypothetical status, is quite interesting for many reasons. First of all, if human or animal brains make use of some sort of quantum computation, a possibility mentioned by Penrose [1] and considered implicitly in the rarely mentioned but pioneering works of Orlov [2], it cannot be excluded that acts of self-observation lead to nonlinear feedback-type effects. Second, the domain of quantum algorithms is an interesting laboratory for testing the concepts and methods developed for the purposes of nonlinear quantum mechanics. The problem which is particularly relevant in this context is the question of how to deal with nonlinearly evolving subsystems of a quantum computer. Quantum computers cannot work without entanglement. To combine entanglement with nonlinear evolution one has to proceed very carefully since it is very easy to produce unphysical effects if one does it wrong. There exists a kind of canonical approach to nonlinearly evolving entangled states but the subject does not seem to be well known to the general audience. The purpose of this paper is to discuss from this perspective two data search algorithms proposed by Abrams and Lloyd [3]. The original version of the algorithms assumed that nonlinear evolution is applied *locally* to a single-qubit system, but the approach used by the Authors is known to generate unphysical nonlocal effects. The version I will describe is free of unphysical influences between different parts of the computer but maintains the essential properties of the algorithms. To make the discussion concrete I use simple albeit somewhat artificial examples of nonlinear Schrödinger equations.

[1] Special Issue on Quantum Optics and Quantum Information

## 2. First algorithm

To make the paper self-contained I will first outline the algorithms. Consider a set $X$ containing $2^n$ elements. We are interested in finding out whether there exist elements $x \in X$ which possess some given property. This is equivalent to defining a function $f : X \to \{0,1\}$ and checking whether there exist $x \in X$ satisfying $f(x) = 1$.

*Step 1.* We begin with the state $|\psi[0]\rangle = |0_1, \dots, 0_n\rangle|0\rangle$, where the first $n$ qubits correspond to the input and the $(n+1)$-th "flag" qubit represents the output, and the unitary transformation

$$U|0\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big), \quad U|1\rangle = \frac{1}{\sqrt{2}}\Big(-|0\rangle + |1\rangle\Big).$$

*Step 2.*

$$|\psi[1]\rangle = \underbrace{U \otimes \dots \otimes U}_{n} \otimes 1|\psi[0]\rangle = \frac{1}{\sqrt{2^n}} \sum_{i_1 \dots i_n = 0}^{1} |i_1, \dots, i_n\rangle|0\rangle.$$

The input consists now of a uniform superposition of all the numbers $0 \le x \le 2^n - 1$.

*Step 3.*

$$|\psi[2]\rangle = F|\psi[1] = \frac{1}{\sqrt{2^n}} \sum_{i_1 \dots i_n = 0}^{1} |i_1, \dots, i_n\rangle|f(i_1, \dots, i_n)\rangle \tag{1}$$

where $F$ is some unitary transformation (an oracle) that transforms the input into an output; $f(i_1, \dots, i_n)$ equals 1 or 0.

*Step 4.* Denote by $s$ the number of occurrences of $f(i_1, \dots, i_n) = 1$.

$$\begin{aligned} |\psi[3]\rangle &= \underbrace{U^{-1} \otimes \dots \otimes U^{-1}}_{n} \otimes 1|\psi[2]\rangle = |0_1, \dots, 0_n\rangle\Big(\frac{2^n - s}{2^n}|0\rangle + \frac{s}{2^n}|1\rangle\Big) \\ &+ \frac{1}{2^n} \sum_{\{j_1 \dots j_n\} \neq \{0_1 \dots 0_n\}} \sum_{i_1 \dots i_n = 0}^{1} (-1)^{(i_1+1)j_1 + \dots + (i_n+1)j_n} |j_1, \dots, j_n\rangle|f(i_1, \dots, i_n)\rangle. \end{aligned} \tag{2}$$

The probability of finding the input in the state $|0_1, \dots, 0_n\rangle$ is $P(s) = \big((2^n - s)^2 + s^2\big)/2^{2n}$. $P(s)$ has a minimum in $s = 2^{n-1}$. The minimal probability of finding the input qubits in $|0_1, \dots, 0_n\rangle$ is therefore $P(2^{n-1}) = 1/2$ and it occurs if $s$ is exactly one-half of $2^n$. Probability of finding $f(i_1, \dots, i_n) = 1$ is $s/2^n$.

*Step 5.* We want to distinguish between the cases $s = 0$ and $s > 0$ for small $s$. To do so we are going to use a nonlinear dynamics that does not change the "North Pole" $|0\rangle$ but any superposition of $|0\rangle$ with $|1\rangle$ drags to the "South". The effect is called, after Mielnik [4], the mobility phenomenon.

## 3. Mobility frequency

Abrams and Lloyd assume that the nonlinear evolution is applied only to the flag qubit and therefore let us first concentrate on a single-qubit system. We shall experiment with different nonlinear equations in order to get some feeling of possible scales of the effects. The first natural try is the equation

$$i|\dot{\psi}\rangle = \epsilon\Big(\frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle} - \langle 0|A|0\rangle\Big)A|\psi\rangle \tag{3}$$

with $A=\eta\Big(|0\rangle\langle 0|-|1\rangle\langle 1|\Big)+\sqrt{1-\eta^2}\Big(|0\rangle\langle 1|+|1\rangle\langle 0|\Big)$ and $\eta$ small but nonzero. The solution of (3) for normalized $|\psi_0\rangle$ is

$$|\psi_t\rangle=\mathbf{1}\cos\Big[\epsilon\Big(\langle\psi_0|A|\psi_0\rangle-\langle 0|A|0\rangle\Big)t\Big]|\psi_0\rangle-iA\sin\Big[\epsilon\Big(\langle\psi_0|A|\psi_0\rangle-\langle 0|A|0\rangle\Big)t\Big]|\psi_0\rangle.$$

Assume

$$|\psi_0\rangle=\frac{2^n-s}{\sqrt{(2^n-s)^2+s^2}}|0\rangle+\frac{s}{\sqrt{(2^n-s)^2+s^2}}|1\rangle.$$

The corresponding mobility frequency is

$$\omega_\epsilon=\epsilon\frac{-2s^2\eta+2(2^n-s)s\sqrt{1-\eta^2}}{(2^n-s)^2+s^2},$$

which for $2^n\gg s$ gives approximately $\omega_\epsilon\approx\epsilon s/2^{n-1}$. This makes the algorithm exponentially slow.

Let us try therefore another nonlinearity:

$$i|\dot\psi\rangle\quad=\quad\epsilon\tanh\Big(\frac{\langle\psi|\psi\rangle}{\langle\psi|A|\psi\rangle}-\frac{1}{\langle 0|A|0\rangle}\Big)A|\psi\rangle. \tag{4}$$

We find

$$\omega'_\epsilon=\epsilon\tanh\Big[\frac{2\eta s^2-2(2^n-s)s\sqrt{1-\eta^2}}{(2^n-s)^2\eta^2-s^2\eta^2+(2^n-s)s\eta\sqrt{1-\eta^2}}\Big]\approx-\epsilon\tanh\Big[\frac{s}{2^{n-1}\eta^2}\Big].$$

For $\eta$ of the order of $1/2^{(n-1)/2}$ one can obtain a reasonable mobility frequency but this requires an exponentially precise control over $\langle 0|A|0\rangle$.

## 4. Evolution of the entire quantum computer

The discussion given above applies to a single-qubit (flag) subsystem. The entire system that is involved consists of $n+1$ systems and therefore we arrive at the delicate problem of extending a one-particle nonlinear dynamics to more particles.

The description chosen by Abrams and Lloyd uses the Weinberg prescription. Several comments are in place here. First, it is known that the Weinberg formulation implies a "faster-than-light telegraph". The version of the telegraph especially relevant in this context is the one that is based on the mobility effect [5]. It is therefore not clear *a priori* to what extent the fact that the algorithm is fast depends on the presence of faster than light effects. Second, the Weinberg prescription is meant to describe systems that do not interact. We have two options now. Either we indeed want to keep the flag qubit noninteracting with the input (during the nonlinear evolution) or we allow a nonlinear evolution which involves the entire quantum computer. If we decide on the first option we should use the Polchinski-type description which eliminates the unphysical nonlocal influences, but the nonlinear evolution of the flag qubit is determined by its reduced density matrix [6 - 11]. This is the reduced density matrix obtained by the reduction over all $2^n$ states of the input subsystem. Physically this kind of evolution occurs if the nonlinearity is active independently of the state of the $n$ input qubits.

But the very idea of the algorithm is to take advantage of the fact that probability of finding the entire input in the ground state exceeds 1/2. It is also assumed that one can turn the nonlinearity on and off. It is legitimate, therefore, to contemplate the situation where the nonlinearity is turned on only provided all the input detectors signal 0.

At this point one might be tempted to act as follows: Take as an initial condition for our nonlinear evolution the product state obtained by projecting the entire entangled state on $|0_1,\dots,0_n\rangle$. The problem with this kind of approach is that the "projection postulate" of linear quantum mechanics does not have an immediate extension to a nonlinear dynamics. There are many reasons for this and the problem is discussed in detail elsewhere [10, 12]. At this moment it is sufficient to know that it is safer to avoid arguments based on the projection postulate if nonlinearity is involved.

I propose an alternative formulation. Assume that indeed the nonlinearity is activated only if the input is in the ground state. In principle there is no problem with this because all the different combinations of 0's and 1's correspond to orthogonal vectors in the $2^n$-dimensional Hilbert space of the input and there exists, in principle, an analyzer that separates the beam of input particles into several different sub-beams. We can place our hypothetical nonlinear medium in front of this output of the analyzer that corresponds to the qubinary zero.

Let us introduce two projectors:

$$P^{(n)} = |0_1,\dots,0_n\rangle\langle 0_1,\dots,0_n| \otimes \mathbf{1}, \quad P = \mathbf{1}^{(n)} \otimes |0\rangle\langle 0|.$$

Denote by $|\Psi\rangle$ the state of the entire quantum computer, $B = \mathbf{1}^{(n)} \otimes A$, and consider the following nonlinear equation

$$i|\dot{\Psi}\rangle = \epsilon \tanh\Big(\frac{\langle\Psi|P^{(n)}|\Psi\rangle}{\langle\Psi|P^{(n)}B|\Psi\rangle} - \frac{\langle\Psi|P^{(n)}P|\Psi\rangle}{\langle\Psi|P^{(n)}PBP|\Psi\rangle}\Big)P^{(n)}B|\Psi\rangle. \tag{5}$$

(5) can be regarded as an appropriate modification of (4). Both expressions occurring under tanh are time-independent. We know that $\Psi_{0_1\dots 0_n 0} = (2^n - s)/2^n$, $\Psi_{0_1\dots 0_n 1} = s/2^n$ and therefore the mobility frequency is identical to the one obtained for a single qubit description. The explicit evolution of the entire entangled state of the quantum computer is finally

$$|\Psi_t\rangle = \Big(\mathbf{1} - P^{(n)} + P^{(n)}\cos\omega'_\epsilon t - iP^{(n)}B\sin\omega'_\epsilon t\Big)|\Psi_0\rangle.$$

The dependence on $P^{(n)}$ reflects our experimental configuration: By changing the projector we change the dynamics since we simply put the nonlinear device in a different position with respect to the first analyzer.

It may be instructive to discuss what would have happened if we had not assumed that the nonlinearity is somehow activated in a state dependent way. We therefore assume that, during the nonlinear evolution, the flag system does not interact with the input one. For this reason we cannot have any dependence on a choice of basis made in the input subsystem, and we use the Polchinski-type extension of (4):

$$i|\dot{\Psi}\rangle = \epsilon \tanh\Big(\frac{\langle\Psi|\Psi\rangle}{\langle\Psi|B|\Psi\rangle} - \frac{\langle\Psi|P|\Psi\rangle}{\langle\Psi|PBP|\Psi\rangle}\Big) B|\Psi\rangle.$$

The solution for the entangled state of our quantum computer is now

$$|\Psi_t\rangle = \Big(\mathbf{1}^{(n+1)} \cos\tilde{\omega}_\epsilon t - iB\sin\tilde{\omega}_\epsilon t\Big)|\Psi_0\rangle,$$

where $\tilde{\omega}_\epsilon$ has to be determined. To do so we first note that the reduced density matrix of the flag subsystem is

$$\mathrm{Tr}_{1\dots n}|\Psi\rangle\langle\Psi| = \frac{2^n - s}{2^n}|0\rangle\langle 0| + \frac{s}{2^n}|1\rangle\langle 1|. \tag{6}$$

The flag subsystem is therefore in a fully mixed state and

$$\tilde{\omega}_\epsilon = \epsilon \tanh\Big(\frac{s}{(2^{n-1} - s)\eta}\Big) \approx \epsilon\tanh\Big(\frac{s}{2^{n-1}\eta}\Big) \ll \omega'_\epsilon$$

so that the algorithm is slower than our previous try.

Returning to the question of exponential precision we should note that the nonlinearity we have chosen leads to periodic dynamics and for this reason has a vanishing Lyapunov exponent. One could invent a nonlinear equation for a two-dimensional dynamics with a positive exponent (cf. [13, 14]) but calculations might be less trivial.

## 5. Second algorithm

We will not follow the original Abrams-Lloyd version but directly describe the modification we have proposed in [15]. The first three steps of the algorithm are identical to the previous ones but in the fourth step we shall use the nonlinear Schrödinger equation

$$i|\dot{\psi}\rangle = \epsilon\tanh\Big(\alpha\langle\psi|A - \eta\mathbf{1}|\psi\rangle\Big)A|\psi\rangle,$$

where $\alpha$ is very large (say, $\alpha \approx 2^n$), and $\eta$, $A$ are as before. For $|\psi\rangle = |0\rangle$ the expression under tanh vanishes. For a small admixture of $|1\rangle$ and sufficiently large $\alpha$ the mobility with a nonzero frequency begins and an arbitrarily small amount of $|1\rangle$ can be sufficiently amplified. The Polchinski-type local extension of the dynamics to the entire quantum computer is

$$i|\dot{\Psi}\rangle = \epsilon\tanh\Big(\alpha\langle\Psi|\mathbf{1}^{(n)}\otimes(A - \eta\mathbf{1})|\Psi\rangle\Big)\mathbf{1}^{(n)}\otimes A|\Psi\rangle.$$

The $(n+1)$-particle solution is

$$|\Psi_t\rangle = \Big(\mathbf{1}^{(n+1)}\cos\omega t - i\mathbf{1}^{(n)}\otimes A\sin\omega t\Big)|\Psi_0\rangle,$$

with
$$\omega = \epsilon \tanh\Big(\alpha \mathrm{Tr}\rho(A-\eta\mathbf{1})\Big) = \epsilon \tanh\Big(\frac{\alpha\eta s}{2^{n-1}}\Big).$$

Here $\rho$ given by (6) is the reduced density matrix of the flag system after the first three steps of the original Abrams-Lloyd algorithm.

The average of $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ at the flag subsystem is

$$\langle\sigma_3\rangle = \langle\Psi_t|\mathbf{1}^{(n)}\otimes\sigma_3|\Psi_t\rangle = \frac{2^{n-1}-s}{2^{n-1}}\cos 2\omega t + 2\eta^2\frac{2^{n-1}-s}{2^{n-1}}\sin^2\omega t.$$

For $s = 0$ the average is constant in time and equals 1. For $s = 1$, $\eta^2 \approx 0$, and sufficiently large $\alpha$ it oscillates with $\omega \approx \epsilon$. For $t \approx \pi/(2\epsilon)$ the average is $\langle\sigma_3\rangle \approx -1$, which means that almost all flag 0's in (1) have been changed to 1's.

This kind of algorithm cannot distinguish between different nonzero values of $s$, but can discriminate between $s = 0$ and $s \neq 0$ in a way that is insensitive to small fluctuations of the parameters.

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