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ENTANGLEMENT AND THERMODYNAMICAL ANALOGIES [1]

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We provide some new properties of entanglement of formation. In particular, we obtain an additive lower bound for entanglement of formation. Subsequently we develop the concept of local orthogonality of ensembles which leads to the mixed states with distillable entanglement equal to entanglement of formation. Then we consider thermodynamical analogies within the entanglement processing domain. Especially, we exploit analogy entanglement – energy. In this scheme the total entanglement i.e. the amount of singlet pairs needed for local preparation of a state corresponds to internal energy while the free entanglement defined as the number of pairs which can be recovered from the state (distillable entanglement) is the counterpart of free energy. In particular, it leads us to the question about "temperature" of entanglement. We also propose a scheme of the search of representative state for given entanglement which can be viewed as an analogue of the Jaynes maximum entropy principle.

1. Introduction

Quantum entanglement discovered by Einstein, Podolsky, Rosen (EPR) [1] and Schrödinger [2], is one of the most interesting quantum phenomena leading to profoundly nonclassical effects revealed by contemporary physics [4, 5, 6, 7, 8]. The entanglement of pure quantum state of composite system is defined as impossibility of factorisation of the state. The simplest and, at the same time, the paradigmatic example of such a state is the singlet state of two spin-$\frac{1}{2}$ particles labelled by A and B

$$\Psi_- = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B), \tag{1}$$

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or any state which can be obtained from the above one by means of product unitary transformation. The properties of the states of this kind are responsible for profoundly nonclassical phenomena like quantum cryptography via Bell inequalities [4], quantum dense coding [5], quantum teleportation [6], quantum computation [7] and the reduction of communication complexity [8]. In real world we usually meet physical systems which interact with environment getting then entangled with it. This process, changing the state of our system from pure to mixed one, described by density matrix, decreases the internal entanglement of the system sometimes even destroying it completely. Thus one usually faces the following undesired process

$$|\Psi_-\rangle\langle\Psi_-| \quad \longrightarrow \quad \varrho_{AB}. \tag{2}$$

Some time ago it has been shown [9, 10, 11] that there are cases when the system in a mixed state ϱ_{AB} still possesses some residual entanglement and physical effects connected with this fact have been discovered [10, 11, 12]. The density matrices of systems possessing residual entanglement are called inseparable. Mathematically, the state ϱ_{AB} defined on the space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is called inseparable (separable) if it cannot (can) be represented as a convex combination of product states

$$\varrho_{AB} = \sum_{i=1}^{K} p_i \varrho_A^i \otimes \tilde{\varrho}_B^i \tag{3}$$

where ϱ_A^i and $\tilde{\varrho}_B^i$ are states of the subsystems and $\sum_{i=1}^{K} p_i = 1$. If the dimensions of the spaces $\mathcal{H}_A$, $\mathcal{H}_B$ are finite then the states ϱ_i and $\tilde{\varrho}_i$ can be taken to be pure and one can consider only the case $K \leq (\dim \mathcal{H}_A \dim \mathcal{H}_B)^2$ [13, 14].

Let us now consider the paradigmatic situation of two observers Alice and Bob being in two distant laboratories. There is a source of pairs of particles between two laboratories which sends one member of any pair to each of them. Alice and Bob are allowed to perform any quantum operations on particles in their laboratories and communicate with each other via some classical channel (say telephone). Usually they are also allowed to discard some particles. We shall refer to all those operations as to LQCC (local quantum operations and classical communication). As Alice and Bob can only interact classically, then some operations are certainly unavailable for them. For example if they share a pair of the particles which are unentangled, it is impossible to entangle those particles with each other. Now the basic task is to find the best Alice and Bob can do under the above restrictions to reverse somehow the process (2). This leads to the recent idea of distillation (or purification) of noisy entanglement via LQCC [15, 16]. In this context, there have been recently some attempts to build an appealing analogy between the domain of entanglement processing and thermodynamics [17, 18, 14, 19]. In this paper we develop the thermodynamics-like approach to entanglement. Especially, we exploit the analogy entanglement – energy. The paper is organized as follows. In Sections 1 and 2 we describe entanglement measures and the process of distillation of entanglement. In Section 3 we show that entanglement of formation may not change under the highly irreversible twirling operation. We also show that there exist mixed states for which the process of local preparing from singlet pairs is reversible i.e. the number of the used pairs can be completely recovered by means of quantum local operations and classical communication. This involves the notion of *local orthogonality.* In Section 4 we describe the analogy entanglement – energy. Then we discuss in this context the known results on entanglement processing. In Section 5 we propose a scheme of obtaining the *representative state* for given value of entanglement. The scheme can be viewed as some analogue of the Jaynes principle of maximum entropy.

2. Entanglement measures

In this section we briefly review some of the recently introduced entanglement measures [25, 22, 14]. Namely there was a natural problem how to quantify the entanglement of any quantum state ϱ_{AB} of composite system. Referring to the paradigmatic situation of two distant laboratories, there arise natural postulates which must be satisfied by any measure of entanglement E [22, 14, 17]

(i) $E(\varrho_{AB})\geq 0$ and $E(\varrho_{AB})=0$ if ϱ_{AB} is separable,

(ii) $E(\varrho_{AB})$ is invariant under the product unitary operations $U_A\otimes U_B$,

(iii) $E(\varrho_{AB})$ cannot be increased by any LQCC operation.

The last axiom should be understood in the stronger averaged entanglement sense i.e. that if we have ϱ^i_{AB} being outputs of some LQCC operation of required property with probabilities p_i then $\sum_i p_i E(\varrho^i_{AB})\leq E(\varrho_{AB})$. We assume here that the considered operation is trace preserving, i.e. it does not involve postselection. In fact, if Alice and Bob selected the subensemble of pairs corresponding some given outcome, then the density matrix of subensemble could be more entangled than the initial state. However, the total average entanglement cannot exceed the initial one. If one adds the condition that

(iv) entanglement of n singlet pairs (1) is equal to n, then one gets that the measure should satisfy [17]

$$E(|\Psi_{AB}\rangle\langle\Psi_{AB}|)=S(Tr_A(|\Psi_{AB}\rangle\langle\Psi_{AB}|),\quad S(\sigma)=-\mathrm{Tr}\sigma\log\sigma, \tag{4}$$

being then *additive* on pure states (in this paper we use base-2 logarithms). Here $S(\sigma)$ is the von Neumann entropy. In general, if we label the Alice and Bob particles by A_i, B_i then the additivity of E is defined as:

$$E(\varrho_{A_1B_1}\otimes\varrho_{A_2B_2}\otimes...\otimes\varrho_{A_nB_n})=E(\varrho_{A_1B_1})+E(\varrho_{A_2B_2})+...+E(\varrho_{A_nB_n}) \tag{5}$$

There are known two measures satisfying all four conditions (i-vi) recalled above. It is not known however whether they are additive. However the condition (iv) automatically guarantees that they are additive on pure states. First one is the entanglement of formation E_f [25] which must be first defined on the quantum ensemble $\mathcal{E}=\{p_i,|\Psi_i\rangle\}$ as

$$E_f(\mathcal{E})=\sum_i p_i S(Tr_A(|\Psi_i\rangle\langle\Psi_i|)). \tag{6}$$

Than the entanglement of formation of state ϱ_{AB} is defined as

$$E_f(\varrho_{AB})=\min_{\mathcal{E}} E_f(\mathcal{E}), \tag{7}$$

where minimum is taken over all ensembles realising the state ϱ_{AB}. The above function has some interesting properties, in particular the analytical formula for E_f for an arbitrary two spin-$\frac{1}{2}$ system has been recently provided [20]. It is not known whether the above function is additive. This is the main obstacle for fully consistent interpretation for this quantity as the asymptotic number of singlets per output pair needed to build given inseparable state. That was the reason of introducing the definition of internal entanglement [28] which we shall call *total entanglement* contained in state ϱ. It is given in the following

Definition.- The total entanglement contained in state ϱ is defined as

$$E_{tot}(\varrho) = \lim_{n\to\infty} \frac{E_f(\varrho^{\otimes n})}{n}, \tag{8}$$

where $\varrho^{\otimes n} = \overbrace{\varrho \otimes ... \otimes \varrho}^{n}$.

This measure has the interpretation of the average number of singlet pairs (per output pair) needed to produce pairs in state ϱ. There is an open question whether the total entanglement is equal to the entanglement of formation. It is certainly *not* greater than the latter and is additive on tensor product of the same states i.e. $E_{tot}(\varrho \otimes \varrho) = 2E_{tot}(\varrho)$. If the entanglement of formation also satisfies this condition then the two measures are identical i.e. $E_{tot} = E_f$. As the problem of additivity of E_f is open, there is the question whether it is possible that $E_f \neq 0$ while $E_{tot} = 0$. Below we provide simple *additive* lower bound for the entanglement of formation which excludes this curiosity for some states.

Proposition.- For any state ϱ_{AB} the entanglement of formation is bounded by:

$$E_f(\varrho_{AB}) \geq S(Tr_C \varrho_{AB}) - S(\varrho_{AB}), \text{ with } C = A \text{ or } B. \tag{9}$$

Proof.- The proof of the above inequality is a simple implication of the concavity property [21] of the function $(S(\cdot) - S(Tr_A(\cdot))$. It is interesting to note that the left hand side of the inequality (9) defines two additive functions of ϱ_{AB}

$$G_C(\varrho_{AB}) = S(Tr_C \varrho_{AB}) - S(\varrho_{AB}), \; C = A \text{ or } B. \tag{10}$$

For 2×2 Werner states any of those functions, if positive, is the amount of entanglement which can be distilled from the states by means of hashing method [25]. It is not known whether this result can be extended to higher dimensions as there is no counterpart of hashing method there. Note that exact positivity of the function $G(\varrho_{AB})$ is *sufficient* condition for inseparability of states. It is important fact that, as we announced before, additivity of $G(\varrho_{AB})$ prevent us from the strange situation when $E_{tot} = 0$ for all inseparable states the inseparability of which results from exact positivity of $G(\varrho_{AB})$. In fact, for $G(\varrho_{AB}) > 0$ we have $E_{tot}(\varrho) = \lim_{n\to\infty} \frac{E_f(\varrho_{AB}^{\otimes n})}{n} \geq \lim_{n\to\infty} \frac{G(\varrho_{AB}^{\otimes n})}{n} = \lim_{n\to\infty} \frac{nG(\varrho_{AB})}{n} = G(\varrho_{AB}) > 0$.

The other measure of entanglement is *relative entropy of entanglement* [22, 14]. This measure is defined as

$$E_r(\varrho) = min_{\sigma_{sep}} S(\varrho||\sigma), \quad \text{with} \quad S(\varrho||\sigma) = Tr(\varrho \log \varrho - \varrho \log \sigma_{sep}). \tag{11}$$

with minimum taken over all separable states σ_{sep}. There is inequality relation between two the measures [14]

$$E_r(\varrho) \leq E_f(\varrho). \tag{12}$$

This measure has good topological properties and important probabilistic interpretation (see Ref. [14]).

3. Distillation of noisy entanglement and notion of distillable entanglement

In this section we describe briefly the concept of distillation of mixed state entanglement. Recall that Alice and Bob start with sharing some amount, say n, of particles in a given initial state and are allowed to perform a sequence of any LQCC operations to obtain with the, in general, less amount (m_n) of pairs of particles in states arbitrarily close to singlets. The asymptotic average amount of the output pairs per input pair is called *distillable entanglement* [15, 25]. Although the main idea is clear, the precise definition of this notion is still missing. An important attempt of strictly mathematical formulation of the definition is contained in Ref.[23] by means of separable superoperators. In that paper it has been proved that using trace preserving separable superoperators, of constant output dimension one can distill from the Werner 2×2 states only *strictly* less amount of entanglement than the entanglement of formation E_f of the states. This important feature of *any* LQCC action has been also derived by means of E_r measure [14] at the additional assumption that the relative entropy of entanglement is additive on product of identical states (at present we do not know whether it is true). As the proof in Ref. [23] does not consider the Alice and Bob actions in full generality, the result is still waiting for rigorous proof. Here we present an attempt the quantitative definition of distillable entanglement [15, 25, 23] enriched by the various dimension output condition. Given n pairs of particles, each in state ϱ, Alice and Bob are allowed to perform arbitrary finite sequences LQCC operations on state $\varrho^{\otimes n}$ with N different final outputs $k = 1, ..., \tilde{M}^n$ with possibly different dimensions d_k^n. Any sequence of LQCC operations can be written as some separable superoperator [23] (but *not* conversely [26]) which can, in general, produce different output systems acting as

$$\varrho^{\otimes n} \to \varrho'_k = \frac{1}{p_i} \sum_{i=1}^{\tilde{N}^n} A_i^{(k)} \otimes B_i^{(k)} \varrho^{\otimes n} {A_i^{(k)}}^\dagger \otimes {B_i^{(k)}}^\dagger, \tag{13}$$

where

$$p_k = \mathrm{Tr}\left(\sum_i A_i^{(k)} \otimes B_i^{(k)} \varrho^{\otimes n} {A_i^{(k)}}^\dagger \otimes {B_i^{(k)}}^\dagger \right) \tag{14}$$

is the probability of the outcome and the states ϱ'_k are defined on different Hilbert spaces H_k^n, dim $H_k^n = d_k^n$; here one also requires that $A_i^{(k)}, B_i^{(k)} : H^{\otimes n} \to H_k^n$ and $\sum_{i,k} p_i A_i^{(k)} A_i^{(k)\dagger} \otimes B_i^{(k)} B_i^{(k)\dagger} = I$.

Now the entanglement which can be distilled by means of such a given protocol $\mathcal{P}$ is defined as

$$D_{\mathcal{P}} = \lim_{n\to\infty} \frac{\overline{m}_n}{n},$$

with $\overline{m}_n = \sum_k p_k \log d_k^n$. Here we demand that the input states ϱ'_k tend to singlet states $\Psi_d^+ = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle \otimes |i\rangle$ on spaces H_k^n for high n. Quantitatively, the closeness between the states and Ψ^+s is measured by fidelity $F = \langle \Psi^+_{d_k^n}|\varrho|\Psi^+_{d_k^n}\rangle$ which should tend to 1 for high n. The latter condition is called *high fidelity condition*. The condition must be stated in such a way, that the input states can be directly used for different purposes like e.g. teleportation. It appears that it is not easy to provide proper form of condition [24] in general case. Of course, for constant output dimension (i.e. where $d_k^n = d_n$ for each k) the condition can be stated unambiguously [25, 23] as

$$\sum_k p_k F(\varrho'_k) \to 1. \tag{15}$$

Also, there is no problem with all the protocols existing so far, as the latter always produce some number of two qubit pairs, so that the output dimensions are powers of 2 × 2. Indeed, suppose that given the outcome k, we obtain m_k^n two-qubit pairs in joint state ϱ'_k. Let ϱ_l^k denote the state of lth pair. The condition can be of the form

$$\lim_{n\to\infty} \inf_{l,k} F(\varrho_l^k) \to 1.$$

Thus we simply demand that the state of each obtained pair has to tend to singlet state for high n. Now, having defined (in somewhat incomplete way) the amount of distilled entanglement with respect to a given protocol we can define [25] the distillable entanglement of the state ϱ by maximizing $D_{\mathcal{P}}$ over all possible protocols $\mathcal{P}$

$$D(\varrho) = \max_{\mathcal{P}} D_{\mathcal{P}}.$$

There are some results on D which are "definition independent" [15, 16, 25, 27]. First, trivially we have $E_{tot} \geq D$ as we certainly cannot obtain greater number of singlets than the one necessary to produce the state. Otherwise we would be able to create singlets by means of LQCC. The inequality immediately implies that also $E_f \geq D$. Much more nontrivial result is that for pure states [16] $D(\Psi) = E_f(\Psi) = E_r(\Psi) = E_{tot}(\Psi)$. It is also known [27] that for any state from 2×2 case $D \neq 0$ iff $E_f \neq 0$ ($E_r \neq 0$). As we have mentioned above there have been provided [23, 14] quite strong arguments supporting the statement that for 2×2 Werner states

$$D < E_f. \tag{16}$$

Quite recently, additional surprising information has been provided [28]. Namely, for the systems $N \times M \geq 8$ there exist states for which $D = 0$ while still $E_f > 0$. The result has been achieved by proving that any distillable state must violate Peres criterion of positivity of partial transposition [29] (see also [30]) and recalling that there are states which satisfy the criterion being still inseparable, having then nonzero E_f, E_r [13]. This result indicates the possibility of dramatic *qualitative* irreversibility of the process production of mixed entangled states. The entanglement needed to produce such states become completely bound, so that no amount of it can be recovered by

$M \times N$ states	
Low dimension - $N \times M \leq 6$	Higher dimension - $N \times M > 6$
$\forall \varrho$ pure $D = E_{tot} = E_f$	
$\forall \varrho$ mixed $E_f > 0 \Rightarrow D > 0$	$\exists \varrho$ (mixed) with $E_f > 0$ and $D = 0$
Probably $\forall \varrho$ mixed $D < E_{tot}$ if only $E_{tot} > 0$	$\exists$ (locally orthogonal) ϱ mixed with $D = E_{tot} = E_f \neq 0$

Tab. 1. Comparison of relations between D and E_{tot}, E_f.

means of LQCC. On the other hand, the inequality (16) represents the *quantitative* irreversibility expected for some mixed states. Since we know that for pure states the process of production of pure not maximal entangled states is reversible ($D = E_f$) then the natural question is whether there are mixed states for which still equality $D = E_f$ holds. We shall discuss this question in detail in the next section. Remarkably, the above irreversibilities, which are physically intuitive, could be regarded as fully exact ones only if we knew that indeed $E_f = E_{tot}$. We do not know it yet, but from the definition of E_{tot} it is easily to show that this quantity is nonzero whenever D is nonzero, and is equal to D if $D = E_f$. In table 1 we collect the results concerning relation between D, E_{tot} and E_f.

4. Local orthogonality concept and mixed states with $D = E_f$

4.1 Illustrative example

As we have mentioned in previous section it has been argued recently [23, 14] that distillable entanglement D is strictly less than the entanglement of formation E for 2×2 Werner states. The latter are of the form

$$\varrho_W(F) \equiv F|\Psi_+\rangle\langle\Psi_+| + \frac{1-F}{3}|\Psi_-\rangle\langle\Psi_-| + \frac{1-F}{3}|\Phi_+\rangle\langle\Phi_+| + \frac{1-F}{3}|\Phi_-\rangle\langle\Phi_-|. \quad (17)$$

Here we use the usual Bell basis [31]

$$\begin{aligned} \Psi_\pm &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle; \\ \Phi_\pm &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle. \end{aligned} \quad (18)$$

Below we provide some quite interesting property of the function E_f which is compatible with the mentioned results. Consider any pure state of the form

$$|\Psi\rangle = a|00\rangle + b|11\rangle, \ a, b > 0, \ a^2 + b^2 = 1. \quad (19)$$

By definition the entanglement of formation of (19) amounts to

$$E_f(\Psi) = H(a^2), \quad (20)$$

where $H(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy function. Let us subject the above state to the following "twirling" operation i.e. the random bilateral operation[5] [9, 25, 15]

$$\mathcal{T}(\cdot) \equiv \int U\otimes U^*(\cdot)U\otimes U^* dU, \tag{21}$$

where the integral is performed with respect to the probability measure proportional to the Haar measure. In result we obtain a Werner state

$$\mathcal{T}(|\Psi\rangle\langle\Psi|) = \varrho_W(F) \quad \text{with} \quad F = \frac{(a+b)^2}{2}. \tag{22}$$

It is elementary to see that $\frac{1}{2} + \sqrt{F(1-F)} = \frac{1}{2}(1+\sqrt{1-4a^2b^2}) = a^2$. Now, since $E(\varrho_W(F)) = H(\frac{1}{2}+\sqrt{F(1-F)})$ [25] we see that

$$E(\mathcal{T}(|\Psi\rangle\langle\Psi|)) = E(\varrho_W(F)) = E(|\Psi\rangle\langle\Psi|). \tag{23}$$

Thus we have the curious situation than any pure entangled state $|\Psi\rangle\langle\Psi|$ has *the same* entanglement of formation as the highly randomized state $\mathcal{T}(|\Psi\rangle\langle\Psi|) = \varrho_W(F)$. Recall that for any pure state $|\Psi\rangle$ one has $D(\Psi) = E(\Psi)$. Thus one can turn all the entanglement contained in Ψ into the pure singlet form. But for the Werner states (17) it has been argued [23, 14] that $D(\varrho_W(F)) \leq \log 2 - H(F) < E_f(\varrho_W(F))$ if only $F<1$. Thus the twirling operation in 2×2 case is probably an example of the operation preserving E_f but significantly decreasing D. In fact, using the Schmidt decomposition one can immediately prove the following

Proposition.- For any 2×2 pure state there exists basis (given by Schmidt decomposition) such that the operation $\mathcal{T}$ defined in this basis preserves entanglement of formation.

For $N\times N$, $N>2$ this proposition is *not* true. To provide the counterexample let us consider $N\times N$ system in the state $\Psi_M = \frac{1}{N}\sum_{i=1}^{M}|ii\rangle$, $1<M<N$ having $E(\Psi_M) = \log M$. After the operation $\mathcal{T}$ one obtains the state [32]

$$\sigma(F) \equiv \frac{N^2}{N^2-1}\left((1-F)\frac{I}{N^2} + (F-\frac{1}{N^2})P_+\right), \quad F = \frac{M}{N}. \tag{24}$$

with $F \leq \frac{M}{N}$, and maximally entangled state $P_+ = \frac{1}{N}\sum_{i=1}^{N}|i\rangle\otimes|i\rangle$. The above state can be represented as $\sigma(F) = \frac{N(1-F)}{N-1}\sigma(\frac{1}{N}) + \frac{NF-1}{N-1}P_+$. As the state $\sigma(\frac{1}{N})$ is separable [32] we get immediately that

$$E(\sigma(F)) \leq \frac{NF-1}{N-1}\log N = \frac{M-1}{N-1}\log N < \log M, \quad \text{for} \quad 1<M<N. \tag{25}$$

Thus in higher dimensions there exist the pure entangled states for which the random bilateral unitary transformation always decreases the entanglement of formation. Thus

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[5] We use here second unitary transformation conjugated relative to the first one for the sake of the further analogy in higher dimensions. For 2×2 case the original twirling operation using the same unitary operations on the first and the second subsystem can be also utilized.

the proposition provides the *unique* feature of two-qubit entanglement. Note in this context that the results concerning distillation of entanglement stress that, in a precise sense, any distillable entanglement is a mixed two-qubit entanglement (see [28] for details).

4.2 Local orthogonality

The expected and partially proved fact that the operation $\mathcal{T}$ decrease D may be supported by the intuition that it should not be possible to distill the full entanglement content from any of pure states $U \otimes U^* \Psi$ included in the mixture $\mathcal{T}(|\Psi\rangle\langle\Psi|)$ as they can not be perfectly distinguished by local operations and classical comunication. Recently this fact has been used in the proof of imposibility of local cloning of Bell states (18) [33]. The essential idea of the latter was the observation that any two orthogonal entangled states can be cloned locally only if some of their reduced density matrices are orthogonal. We shall call this property *local orthogonality* proposing the general definition

Definition.- Consider two states ψ, ϕ of composite multiparticle composite quantum system defined on Hilbert space $\mathcal{H} = \bigotimes_{l=1}^{m} \mathcal{H}_l$. We say that the two states are **k-locally orthogonal**, *if there exist some k subsystems (say they are labelled by $\{i_1, \ldots, i_k\}$ and are described by Hilbert spaces $\mathcal{H}_{i_1}, \ldots, \mathcal{H}_{i_k}$) such that the corresponding reductions of the states ψ, ϕ are orthogonal, i.e.*

$$Tr(\varrho_l^{\psi} \varrho_l^{\phi}) = 0, \quad l = i_1, ..., i_k. \tag{26}$$

If the numbers of the systems $\{i_1, \ldots, i_k\}$ are known then the states ψ, ϕ are called **locally orthogonal on the subsystems** *$i_1, \ldots, i_k$. The ensemble of pure states $\{\psi_i\}_{i=1}^{K}\}$ is called* **locally orthogonal** *if its elements can be ordered in the sequence $\{\psi_{i_1}, \psi_{i_2}, ..., \psi_{i_K}\}$ such that for any $1 \leq m \leq N$ the state ψ_{i_m} is 1-locally orthogonal on the same subsystem to all following states $\psi_{i_n}, n > m$.*

Remark.- The notion of local orthogonality is not equivalent to the notion of local distinguishability. The states in a given ensemble can be locally distinguishable, but it may be the case, that to distinguish them one must destroy them (see in this context [26]). The locally orthogonal states are distinguishable without destroying them.

Note that any two component quantum system allowing for existence two *entangled* 1-locally orthogonal pure states must be of the form $M \times N$, $\max(M, N) \geq 4$ i.e. one of subsystems must represent at least the spin $\frac{3}{2}$. We have the following simple

Property.- Consider the state $\varrho = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|$ of the quantum system composed from two subsystems defined by locally orthogonal ensemble $\{\Phi_i\}$. Then we have

(i) $E_f(\varrho) = \sum_i p_i E_f(|\Phi_i\rangle\langle\Phi_i|)$

(ii) $D(\varrho) = E_f(\varrho) = E_{tot}(\varrho)$.

Proof.- Consider the system in state ϱ in paradigmatic situation of two distant laboratories and let us treat the state ϱ as a random mixture of Φ_is. Then, using appropriate local measurements (following from local orthogonality property of the ensemble) and classical communication Alice and Bob can determine which of the pure state they share. Then they can use the large blocks procedure [16] to convert *all* the entanglement of this state into the singlet form D. Thus we have $D = \sum_i p_i E_f(|\Phi_i\rangle\langle\Phi_i|)$. As, by definition, $D(\varrho) \leq E_f(\varrho) \leq \sum_i E_f(|\Phi_i\rangle\langle\Phi_i|)$ we obtain both numerical value of E_f and its equality to D. As the latter is, again by definition, additive on products of the same states and as $E_{tot} \leq E_f$ we get easily from the first equality of (ii) the second one.

The simple example of 5×5 locally orthogonal ensemble ordered as required in the definition is: $\{|00\rangle + |11\rangle, |21\rangle + |32\rangle, |30\rangle + |43\rangle$. There remains an interesting question of maximal support of the locally orthogonal ensembles for fixed composite quantum system. In the above context we propose the two following conjectures:

Conjecture 1.- The locally orthogonal ensembles are the only ones which pure entangled components Ψ_i can be distinguished by local operations and classical communication without destroying them.

Conjecture 2.- The only mixed states ϱ with the property $E_{tot}(\varrho) = D(\varrho)$ are those which are defined by means of locally orthogonal ensembles.

Finally, note that the result of providing mixtures with $D = E_{tot}$ is analogous to that of Braunstein, Mann and Revzen [31] who found mixtures violating maximally the Bell inequalities. Their mixtures were, in our language, locally orthogonal mixtures of singlet pairs.

5. Thermodynamical analogies

The first formal thermodynamical analogy was proposed by Popescu and Rohrlich [17]. The authors made an important observation that any LQCC process which preserves entanglement must be *reversible* which have been related to Carnot cycle in usual thermodynamics. Such an analogy was also considered by Vedral and Plenio [14] in the context of distillation of entanglement. In more general quantum information context the thermodynamical approach was also developed in Ref. [18] where, in particular, the law of conservation of quantum information (entanglement) was considered. The analogy entanglement-energy was then developed in Ref. [28] to interpret the results on distillation of entanglement. Below we propose some new elements which, we believe, will contribute to understanding of thermodynamics-like aspects of entanglement processing.

From previous discussion we know that there are two important measures of entanglement which are in a sense dual ones:

- Total entanglement $E_{tot}(\varrho)$ which represents the least number of shared singlets asymptotically required to prepare the state ϱ by means of LQCC.
- Distillable entanglement $D(\varrho)$ - the greatest number of pure singlets that can asymptotically be prepared from ϱ by means of LQCC.

In short, E_{tot} is the minimal number of singlets we need to produce a state while D is the maximal number of singlets we can recover from the state. Another important quantity of precise information theoretic sense is von Neumann entropy S. Its physical sense in quantum information theory was proved by Schumacher [34] (see also Ref. [35]) and Barnum *et al.* [36] in the context of compressing of quantum information. Basing on the above notions we shall now try to provide information-theoretic counterparts of such thermodynamical notions as internal energy U, free energy F and entropy S_{th}. We propose the following scheme for the "energetic" quantities [28] E_{tot}, D.

$$\text{(free entanglement)}\quad D = E_{free} \leftrightarrow F \quad \text{(free energy)}$$

$$\text{(total entanglement)}\quad E_{\text{tot}} \leftrightarrow U \quad \text{(internal energy)}$$

$$\text{(bound entanglement)}\quad E - D \equiv E_{\text{bound}} \leftrightarrow TS_{th} \quad \text{(bound energy)}$$

where T is temperature. Now the last element of the above scheme lead us to the question what about the entanglement counterparts of thermodynamical temperature and entropy? Following analogy we obtain a formula

$$E_{tot} = E_{free} - T_e S_e, \tag{27}$$

where T_e and S_e are unknown counterparts of T and S_{th}.

If one tries to recognize the von Neumann entropy S as the counterpart of thermodynamical entropy then finding the interpretation of temperature of entanglement T_e would be the main test of this choice. It can be partially verified that the choice might be reasonable. First, note that in thermodynamics one has $F_{th} \leq U_{th}$. In our case the analogous inequality holds, as trivially, D cannot exceed E_{tot}. Now, recall that in usual thermodynamics $F = U - TS_{th}$. According to our proposal $E_{free} = E_{tot} - T_e S$ even if we do not know what the temperature T_e means. Suppose now, that for pure states T_e is finite. Then putting $S = 0$ we obtain that $D = E_{tot}$, i.e. that for pure states, the distillable entanglement is equal to the entanglement of formation, which is true indeed [16]. Note, that for mixed states, the formula (27) can serve as a definition of temperature T_e for mixed states ϱ:

$$T_e(\varrho_{mix}) = \frac{E_{tot} - E_{free}}{S}. \tag{28}$$

As for separable mixed state ϱ we have $E = D = 0$ we obtain that the temperature vanishes in this case which can be extrapolated via expected continuity property to cover the case of pure separable states. So far the von Neumann entropy seems to be a good quantity for our purposes. But, as we shall see in a moment, there are some problems. Note that for many mixed inseparable states, the best known protocols of distillation provide a very little number of distilled singlets in comparison with the entanglement of formation. For example for Werner states the obtained yield of distillation procedure is about thousand times less than the entanglement of formation. There are even more stronger suggestions (recalled in the previous sections) that for mixed 2×2 inseparable states $E_{free} < E_{tot}$ (provided that $E_{tot} = E_f$). Now, in our language, this is equivalent to state that for those states the temperature is nonzero. However, as we have shown in Section 4 for composite systems of higher dimensions there are many states with $E_f = E_{tot} = E_{free}$ and $S \neq 0$. This is a curiosity of the model and it would suggest that the temperature of those states is zero which is not intuitive. Thus perhaps the entropy S_e should be defined in other way. For this purpose one could exploit the notion of local orthogonality. Namely, the entanglement entropy S_e should quantify the irreversibility of the process of local preparing of the given mixed state. Then it could be defined as some measure of local non-orthogonality of some *canonical* ensemble realizing the state. A natural candidate is here the optimal ensemble in formula defining the entanglement of formation. So defined entropy would vanish for locally orthogonal mixtures. However, it is a very hard task to obtain a precise definition, and it is an open question, whether that choice would produce the entanglement temperature having a good physical interpretation. Nevertheless, the energy type analogies about D as a counterpart of free energy and E_{tot} as the counterpart of internal energy seem to be plausible.

The remaining question is to define the entanglement (informational) work. In general, it is natural to assume that

$$\text{sending qubits} \leftrightarrow \text{WORK}.$$

Indeed, suppose that the given system contains some amount of free entanglement. Then one can use it to send precisely this amount of qubits by means of teleportation preceded by distillation procedure. This is analogous to the situation in thermodynamics, where the free energy can be used to perform work. On the other hand, if the work ΔW is performed over a system (without dissipation i.e. adiabatically) then the total energy increases $U_2 = U_1 + \Delta W$. This corresponds to the fact that to produce a state of total entanglement E_{tot} Alice and Bob need to exchange precisely this amount of qubits (they will send halves of singlets). This is true if the channel between Alice and Bob is noiseless which correspond to lack of dissipation of energy. If, instead, the channel is noisy, then the number of qubits must be larger: some amount of the sent entanglement will be *spread* over the system and environment [6]. Then, despite the total entanglement (total quantum information) of system plus environment is conserved, the amount of *useful* entanglement is much more less than the number of exchanged qubits. We may say that there is the "information heat flow": the lost entanglement was not used to perform work, but rather changed into uncontrolled form. In the context of the possibility of existence of states with nonzero bound entanglement, the process of this flow can manifest itself not only by destroying some amount of entanglement but also by binding some part of remaining entanglement. The above consideration is nothing else but a balance analogous to that governed by the first law of thermodynamics. As the latter is nothing else but the principle of conservation energy, we obtain that the above balance is implied by the conservation of quantum information which can be viewed as the analogue of the thermodynamical law for quantum entanglement processing [18].

The reader can ask why sending qubits and sending entangled qubits can both be treated as work. Indeed there is a clear qualitative difference between them. However, it can be explain as follows: sending *known* entangled qubits can be represented as work *over* the composite system i.e. as a counterpart of mechanical work done over the gas, as it aims at increasing of entanglement - "energy" of the system. The latter can be

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[6] Note that entanglement can be spread deliberately by means of a kind of "depurification" procedure [37].

subjected to dissipation which causes the information heat flow discussed above. On the other hand, having the composite system with given entanglement E_{free} one can teleport some unentangled qubit through it. This represent the *work done by the system* for us, resembling the gas doing mechanical work. One can also send the unentangled qubits directly without using entanglement and teleportation. Then for noiseless channel we have a counterpart of purely mechanical process (with no thermodynamical element).

6. Searching for representative state at given entanglement

In this section we consider the problem of choosing for each established value of entanglement, some representative, most probable state. Such a choice can be of course performed only up to local unitary transformation. As a criterion we will use von Neumann entropy, so that the scheme of obtaining the representative state will slightly resemble the Jaynes scheme [38] of producing Gibbs state[7]. Recall that given the Hamiltonian H the Gibbs state $\varrho_G = (1/\mathrm{Tr}e^{\beta H})e^{\beta H}$ for a given mean energy $E = \langle H\rangle = \mathrm{Tr}\varrho H$ can be obtained by maximizing von Neumann entropy over all states with mean energy E. In our scheme, we will keep constant entanglement and maximize entropy (see [19] in this context). As it is total entanglement which we chose as the counterpart of energy, we should use this measure in our scheme. However, to be able to perform any calculations we must have analytical formula for entanglement. In this situation we will rather use entanglement of formation. In the case of two-qubit states the analytic formula for the latter is the following [20]

$$E_f(\varrho) = H(\frac{1+\sqrt{1-C^2}}{2}). \tag{29}$$

Here $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ and λ_is are the eigenvalues, in decreasing order, of the Hermitian matrix $R = \sqrt{\sqrt{\varrho}\tilde{\varrho}\sqrt{\varrho}}$ with

$$\tilde{\varrho} = \sigma_y \otimes \sigma_y \varrho^* \sigma_y \otimes \sigma_y \tag{30}$$

The star denotes complex conjugation of the matrix ϱ in product basis. Now, one could ask what state is expected to have the greatest von Neumann entropy of all states with given entanglement of formation. Such a state should have a high degree of symmetry. The natural candidate is the Werner state (17) as it can be written in the following very symmetric form

$$\varrho_W = \alpha|\Psi_+\rangle\langle\Psi_+| + (1-\alpha)\frac{1}{4}I \otimes I; \qquad \frac{1}{3} \leq \alpha \leq 1, \tag{31}$$

where $\alpha = \frac{1}{3}(4F-1)$. Surprisingly, we will see that for some values of E_f the representative state is certainly not the Werner state. Consider the following simple state

$$\varrho_p = p|\Psi_+\rangle\langle\Psi_+| + (1-p)|01\rangle\langle 01|. \tag{32}$$

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[7] Note that in entanglement processing domain the inference scheme based on Jaynes principle in general fails [19].

It can be checked that it is entangled for any $p > 0$ [25, 29, 30]. We obtain that

$$C(\varrho_p) = p, \quad C(\varrho_W) = 2F - 1. \tag{33}$$

Then the two states have the same entanglement of formation if $p = 2F - 1$. This is compatible with the fact that for $p = F = 1$ both of them are equal to $|\Psi_+\rangle\langle\Psi_+|$ while for $p = 0, F = \frac{1}{2}$ both of them are separable. Now, one can check that at least for $\frac{3}{4} < p < 1$ the entropy of the state ϱ_p is strictly greater than the one the Werner state. Consequently, it is certainly not the Werner state which maximizes entropy for those values of entanglement. Then one can conclude that the problem of finding the representative state for given entanglement may produce some highly nontrivial result: it will provide an interesting, unknown family of states.

7. Concluding remarks

In conclusion, we have provided new properties of entanglement of formation. In particular an additive lower bound for this quantity has been provided. The concept of local orthogonality have been developed leading to the family of mixed states with distillable entanglement equal to entanglement of formation. Subsequently the notions of total and distillable entanglement have been considered as counterparts of thermodynamical notions of internal and free energy. The question of possible temperature of entanglement as well as a counterpart of thermodynamical entropy have been analysed. The process of sending quantum information has been considered as a counterpart of work and discussed in detail.

One of the advantages of the proposed approach is certainly the fact that it generates new interesting questions like e.g. the problem of defining and interpretation of temperature of entanglement. Another, fundamental problem is the following: is there a link between the analogy considered here and the recent development concerning quantum information processing at incomplete data [19, 39]? In fact, since the famous Jaynes papers we know that the statistical thermodynamics which explains the phenomenological one can be treated as a special kind of statistical inference at incomplete experimental data. Then, it follows that the recent results on quantum information processing at incomplete data (where one requires some nonstandard schemes [19]) should also be somehow connected with the approach discussed in this paper. We think that since analogy was always a powerful tool in physics, the above problems are worth of deeper investigation. We believe that the present consideration will contribute to obtaining more clear picture of the highly nonintuitive domain which is quantum information theory.

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