:ta physica slovaca vol. 48 No. 1, 23 – 42 February 1998

NUMBER-PHASE WIGNER REPRESENTATION AND ENTROPIC UNCERTAINTY RELATIONS FOR BINOMIAL AND NEGATIVE BINOMIAL STATES

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Received 5 December 1997, in final form 8 January 1998, accepted 12 January 1998

We study the recently defined number-phase Wigner function $S_{NP}(n,\theta)$ for a single-mode field considered to be in binomial and negative binomial states. These states interpolate between Fock and coherent states and coherent and quasi-thermal states, respectively, and thus provide a set of states with properties ranging from uncertain phase and sharp photon number to sharp phase and uncertain photon number. The distribution function $S_{NP}(n,\theta)$ gives a graphical representation of the complimentary nature of the number and phase properties of these states. We highlight important differences between Wigner's quasiprobability function, which is associated with the position and momentum observables, and $S_{NP}(n,\theta)$, which is associated directly with the photon number and phase observables. We also discuss the number-phase entropic uncertainty relation for the binomial and negative binomial states and we show that negative binomial states give a lower phase entropy than states which minimize the phase variance.

1. Introduction

Wigner's function $W(x,p)$ [1] has been used extensively in quantum optics for solving dynamical problems in phase space as well as for studying the nature of nonclassical states of light. For example, the Wigner function gives a graphic representation of the relationship between the uncertainties in the quadrature amplitudes of the field for squeezed states. It has also been used to illustrate quantum decoherence where the decay of the coherence of a macroscopic superposition state (a Schrödinger-cat state) is represented by the decay of interference fringes in $W(x,p)$ [2].

Wigner's function is defined with respect to the canonically conjugate position and momentum observables, or equivalently, the quadrature amplitude observables of the single-mode electric field in the optical case. In classical mechanics, however, one can

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also give a phase-space description of the single-mode field in terms of the intensity and the phase as well as in terms of the quadrature amplitudes. A great deal of attention has been given in the last few years to the definition of an acceptable quantum-mechanical counterpart of classical phase and there are now a number of mathematically-different formalisms [3, 4, 5, 6, 7, 8, 9, 10] available that give a mutually-consistent physical description of quantum-mechanical phase. An important feature of this common description is that it defines quantum phase as being canonically conjugate to photon number [11]. The advent of this well-defined description recently led one of us [12, 13] to define a quasi-probability distribution which has properties analogous to those of the original Wigner function but which is associated with the canonically conjugate phase and photon number observables [14] rather than the position and momentum observables used by Wigner. This quasiprobability distribution, which we shall call here the number-phase Wigner function $S_{NP}(n,\theta)$, is defined on the infinite-dimensional Hilbert space which supports both the canonical phase and photon number probability distributions. The $S_{NP}(n,\theta)$ representation of Fock states, coherent states, squeezed states and Schrödinger cat states (in terms of even and odd coherent states) has been studied in Ref. [12, 13] and many interesting features related to the phase and photon number properties of these states have been discussed. While for some states, e.g. intense coherent states, both $W(x,p)$ and $S_{NP}(n,\theta)$ were found to be qualitatively similar, for other states such as the squeezed vacuum they were found to differ significantly. The difference can be traced to the different variables $\hat{x}$, $\hat{p}$ or $\hat{N}$, $\hat{\phi}$ upon which the functions are based. This difference allows $S_{NP}(n,\theta)$ to give an alternate phase-space description of quantum states which is associated directly with their number and phase properties.

In this work we study the nature of $S_{NP}(n,\theta)$ for a single-mode field prepared in a binomial or a negative binomial state. Binomial and negative binomial states have been studied well previously and a possible method for their experimental generation has also been proposed [15, 16, 17, 18, 19]. Binomial states display antibunching and sub-Poissonian statistics as well as squeezing. The dynamics of a field prepared initially in one of these states interacting with a two-level atom has been reported in the literature within the framework of the Jaynes-Cummings model and many interesting effects relating to collapse-revival phenomenon of Rabi oscillations have been obtained [17, 19]. A binomial state is "intermediate" between a Fock and a coherent state in the sense that the Fock and coherent states are two different limiting cases of binomial states [18]. Thus by varying an appropriate parameter (e.g., photon number variance) for fixed mean photon number a binomial state interpolates between a state with random phase (Fock state) to one with sharply defined phase (coherent state). Similarly a negative binomial state is "intermediate" between a coherent state and a quasi-thermal state [18] and thus a negative binomial state interpolates between a coherent state and a state which have even more sharply-defined phase. Moreover both binomial and negative binomial states can exhibit squeezing which is a phase sensitive effect. Thus the photon number and phase properties of these states is clearly important and the $S_{NP}(n,\theta)$ function allows us to study these properties in a vivid graphical manner.

We also study in this paper the number-phase entropic uncertainty relations for binomial and negative binomial states. Entropic uncertainty relations [20, 21, 22, 23] act as an alternative to the more familiar Heisenberg uncertainty relations [24]. They give a finite lower bound on the entropic uncertainty associated with pairs of observables, which, in our case, are the number and phase observables. In general, different approaches are required to treat the problem depending on whether the spectrum of eigenvalues for the operator considered is continuous or discrete and bounded or unbounded. Recently Rojas González *et. al.* [25] presented a unified approach for treating all types of operators. For this purpose the general method for formulating canonically conjugate operators given by Pegg *et. al.* [26] was used. This method allows one to derive the entropic uncertainty relation for an infinite dimensional system from the general form of the relation for finite dimensions. The number-phase entropic uncertainty relation was used recently in a study of the emergence of Schrödinger cat states in a field interacting with a Kerr medium [27]. The minimisation problem of finding the number-phase uncertainty state is considered in [28] and the minimum uncertainty state relations between the photon number uncertainty and the phase uncertainty are presented. A very interesting description of quantum mechanical states based on operational approach to a phase space measurement are studied by Buzek *et. al.* [29]. These authors[29] have derived general entropic uncertainty relations which reflects the phase space uncertainty of the quantum mechanical state in a given measurement and they have illustrated it with many examples. Shanon entropy of position and momentum for the stationary quantum states of the harmonic oscillator as a function of its energy is calculated and corresponding entropic uncertainty relations for them are determined along with some examples[30]. Very recently, phase-intensity uncertainty relation from quasiprobability distribution are reported by Orlowski *et al.*[31]. We study here the number-phase entropic uncertainty relation for binomial and negative binomial states and look at how the number and phase entropies change as these states interpolate between Fock and coherent states and between coherent and quasi-thermal states, respectively. Our study complements and extends recent work by Gantsog *et. al.* [32] who examined the number-phase variances of these states. In particular, we compare the phase entropy and phase variance of binomial and negative binomial states with that of Airy states [33, 34]. Airy states are important because they approximate states with the minimum phase variance for given intensities and thus they give, approximately, the lower bound on the phase variance.

The paper is organized as follows. In section 2 we give a summary of the statistical properties of the binomial and negative binomial states. In section 3 we use the number-phase Wigner function $S_{NP}(n,\theta)$ to illustrate graphically the photon number and phase properties of these states and in section 4 we discuss the number-phase entropic uncertainty relation and examine the associated number and phase entropies. We end with a discussion in section 5.

2. Binomial and Negative binomial states: a brief review

Binomial state

The binomial states are defined as [15]

$$|p, M\rangle = \sum_{n=0}^{M} C_n(p)\,|n\rangle, \tag{1}$$

where

$$C_n(p) = [\frac{M!}{n!\,(M-n)!}\,p^n\,(1-p)^{M-n}]^{1/2}, \quad n = 0,1,2..M, \quad 0 \le p \le 1, \tag{2}$$

which means that the binomial state describes the state of the field having a binomial distribution of photon number with a mean of $\bar{n} = Mp$, and a variance of $(\Delta n)^2 = \overline{n^2} - \bar{n}^2 = Mp(1-p)$. Since the variance is always less than the mean the Mandel parameter $Q = [(\Delta n)^2/\bar{n}] - 1$, which signifies the deviation from the Poissonian distribution, is always negative. In other words the photon statistics is always sub-Poissonian. For $p = 1$, $C_n(p) = \delta_{n,M}$ and binomial state reduces to the Fock state $|M\rangle$. In other extreme, for $p \to 0$ and $M \to \infty$ but with $Mp = \bar{n}$ held constant, the binomial state becomes a coherent state with mean number of photon $\bar{n}$.

Negative Binomial state

The negative binomial states are defined as [18, 19]

$$|q, w\rangle = \sum_{n=0}^{\infty} C_n(q, w)\,|n\rangle, \tag{3}$$

where

$$C_n(q, w) = [\frac{(n+w)!}{n!\,w!}\,q^n\,(1-q)^{w+1}]^{1/2}, \tag{4}$$

with $w \ge 0$, $0 \le q \le 1$, $n = 0, 1, 2, \ldots, \infty$. The probability of finding n photons in state (3) is given by the negative binomial distribution

$$P_n = \frac{(n+w)!}{n!\,w!}\,q^n\,(1-q)^{w+1}, \tag{5}$$

whose mean and variance are given by

$$\bar{n} = (1+w)\frac{q}{1-q}, \quad (\Delta n)^2 = (1+w)\frac{q}{p^2}. \tag{6}$$

The Mandel Q parameter for the negative binomial state is given by

$$Q = [(\Delta n)^2/\bar{n}] - 1 = \frac{q}{1-q}, \tag{7}$$

which is always positive since $0 \le q \le 1$. This implies that the photon statistics of the negative binomial state is always super-Poissonian. For $w \to 0$, the photon number distribution reduces to the Bose-Einstein distribution with a mean of $\bar{n} = \frac{q}{1-q}$ and the negative binomial state becomes a quasi-thermal state. In the other limit for $w \to \infty$, $q \to 0$, but with $\bar{n} = (1+w)q/(1-q)$ held constant, the photon number distribution reduces to the Poissonian distribution and the negative binomial state becomes the coherent state $|\alpha\rangle$ with $\alpha = \sqrt{\bar{n}}$.

3. The number-phase Wigner function

The number-phase Wigner function $S_{NP}(n, \theta)$ is defined as the expectation value of the number-phase Wigner operator $\hat{S}_{NP}(n, \theta)$ which is given by [12, 13]

$$\hat{S}_{NP}(n,\theta) = (2\pi)^{-1}\left\{\sum_{k=-n}^{n} \exp(2ik\theta)|n+k\rangle\langle n-k| + \sum_{k=-n}^{n-1} \exp[2i(k+1)\theta]|n+k\rangle\langle n-k-1|\right\}, \tag{8}$$

in the Fock basis for $n = 0, 1, 2, \ldots..$ and θ real. The second sum on the right-hand side is defined to be zero for $n = 0$. Alternatively, this operator may be expressed as

$$\hat{S}_{NP}(n,\theta) = (2\pi)^{-1}\int_{2\pi} \exp(-2in\theta)[1+\exp(i\phi)]|\theta+\phi\rangle\langle\theta-\phi|d\phi, \tag{9}$$

where $|\theta\rangle$ is a phase state [35, 4, 5, 7, 9] which belongs to the rigged Hilbert space [36] and is defined by

$$|\theta\rangle = (2\pi)^{-1}\sum_{n=0}^{\infty} \exp(in\theta)|n\rangle. \tag{10}$$

We note that $|\theta\rangle$ is the weak limit of the Pegg-Barnett phase state [5]. It is straightforward to check that these two expression for $S_{NP}(n, \theta)$ are equivalent by taking matrix elements of (9) in the Fock basis and comparing with the corresponding matrix elements in (8).

The position-momentum Wigner function [1] defined as

$$W(x,p) = (\pi)^{-1}\left\langle \int_{-\infty}^{\infty} \exp(-2ixy)|p+y\rangle\langle p-y|dy \right\rangle, \tag{11}$$

in which $|p\rangle$ is a momentum eigenstate, is formally similar to $S_{NP}(n, \theta)$. Comparing (9) and (11) we find that apart from the different limits of integration, there is the extra factor $1 + \exp(i\phi)$ in (9). The origin of this factor is due to the discrete nature of photon number observable [13].

The number-phase Wigner function has a number of defining properties which are directly analogous to those of the position-momentum Wigner function [13]. Those that are important for us here are that $S_{NP}(n,\theta)$ is real and its marginal distributions are the normalized number and phase probability distributions, P_n and $P(\theta)$ respectively. These distribution are given as follows:

$$P_n = \int_{2\pi} S_{NP}(n,\theta)d\theta \tag{12}$$

$$P(\theta) = \sum_{n=0}^{\infty} S_{NP}(n,\theta) \tag{13}$$

where $P_n = |\langle n|f\rangle|^2$, $P(\theta) = |\langle\theta|f\rangle|^2$ and $S_{NP}(n,\theta) = \langle f|\hat{S}_{NP}(n,\theta)|f\rangle$ for arbitrary state $|f\rangle$.

We now use the $S_{NP}(n,\theta)$ defined above to study quantum properties of the binomial and negative binomial states of the field mode.

Binomial state

The number-phase Wigner function $S_{NP}(n,\theta)$ for the binomial state (1) is given by

$$S_{NP}(n,\theta) = (2\pi)^{-1}\left\{\sum_{k=-n}^{n} \frac{\exp(2ik\theta)p^n(1-p)^{M-n}M!}{[(n+k)!(n-k)!(M-n-k)!(M-n+k)!]^{1/2}} + \sum_{k=-n}^{n-1} \frac{\exp(2i(k+1)\theta)p^{n-1/2}(1-p)^{M-n+1/2}M!}{[(n+k)!(n-k-1)!(M-n-k)!(M-n+k+1)!]^{1/2}}\right\} \tag{14}$$

for $0 \le n \le M$ and $S_{NP}(n,\theta) = 0$ otherwise. Here, and in the remainder of this paper, we take the parameter p to be real and non-negative. For the case of reference we give $S_{N,P}(n,\theta)$ for both the Fock and coherent states as follows. The $S_{N,P}(n,\theta)$ for the Fock state $|M\rangle$ is

$$S_{N,P}(n,\theta) = (2\pi)^{-1}\delta_{n,M}, \tag{15}$$

which is represented graphically simply as a raised ring of radius M above the (n,θ) plane. The phase is completely random whereas the photon number is sharp. For the coherent state $|\alpha\rangle$ with α real and positive we find

$$S_{NP}(n,\theta) = \frac{\exp(-|\alpha|^2)}{(2\pi)}\left\{\sum_{k=-n}^{n} \frac{\exp(2ik\theta)|\alpha|^{2n}}{[(n+k)!(n-k)!]^{1/2}} + \sum_{k=-n}^{n-1} \frac{\exp(i(2k+1)\theta)|\alpha|^{2n-1}}{[(n+k)!(n-k-1)!]^{1/2}}\right\}, \tag{16}$$

which is approximated by a two-dimensional Gaussian as follows [13]

$$S_{N,P}(n,\theta) \approx (2\pi)^{-1}\exp[-2|\alpha|^2\theta^2]\exp[-(n-|\alpha|^2)/2|\alpha|^2]$$

for $\alpha \gg 1$.

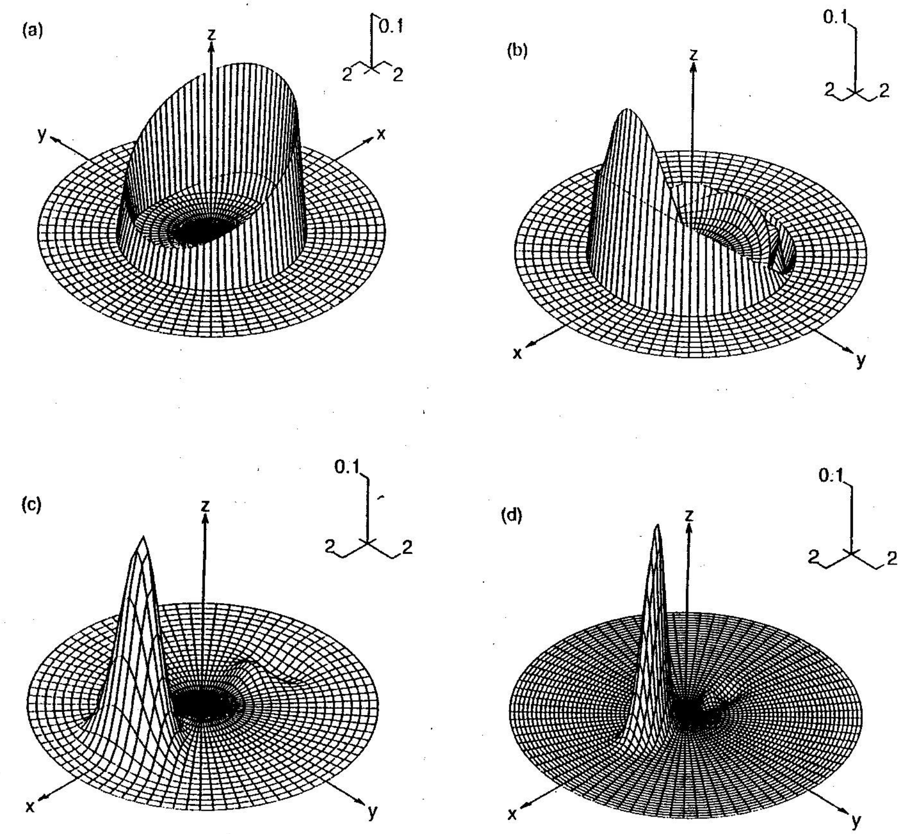

Fig. 1. The $S_{NP}(n,\theta)$ representation of the binomial state $|p,M\rangle$ with $\bar{n} = pM \approx 10$ for various values of M and p. $S_{NP}(n,\theta)$ is represented in cylindrical coordinates as the "surface" $z = S_{NP}(n,\theta)$ above the points (x,y) with $x = n\cos(\theta)$ and $y = n\sin(\theta)$. The triad in the upper right corner gives the axis scaling in arbitrary units (a.u.). The surface is drawn as curves of constant n (concentric rings incremented by unit one) crossed by curves of constant θ (radial lines). The representation in (a) for $M = 10$, $p = 0.99$ approaches that of the Fock state $|10\rangle$. In (b) $M = 11$, $p = 0.9$ and (c) $M = 20$, $p = 0.5$ the binomial state clearly displays nonrandom phase and a spread in photon number. In (d) with $M = 1000$, $p = 0.01$ the representation approaches that of the coherent state $|\alpha\rangle$ with $|\alpha|^2 = 10$.

Since binomial states interpolate between Fock and coherent states one would expect that $S_{N,P}(n,\theta)$ in (14) will give a picture which interpolates between the completely random phase of the number state and the relatively sharply defined phase properties of a coherent state. This is indeed the case as shown in Fig.1 in which we plot $S_{NP}(n,\theta)$ for

$\bar{n} = pM \approx 10$ with p varying from 0.99 to 0.01. It can be seen that near $p = 1$ (Fig.1(a), $p = 0.99$) the features of $S_{NP}(n,\theta)$ resembles that of the $S_{NP}(n,\theta)$ representation of a Fock state, i.e., the phase is nearly random and the photon number is relatively sharply defined. As we decrease p (Fig.1(b)-(d)) the phase becomes less random and the photon number becomes more uncertain. In Fig.1(b) with $p = 0.9$ the phase is already far from random although it still has a broad distribution centered on $\theta = 0$ and the photon number has spread over a few integer values. This trend continues for $p = 0.5$ (Fig. 1(c)) and $p = 0.01$ (Fig.1(d)). In particular $S_{NP}(n,\theta)$ in Fig.1(d) is quite similar to that of coherent state $|\alpha\rangle$ with $\alpha = \sqrt{10}$.

The $S_{NP}(n,\theta)$ representation of the binomial states in these figures gives a clear illustration of the number-phase complementarity: a decrease in the phase uncertainty is accompanied by an increase in number uncertainty and vice versa. This situation can be contrasted with the picture obtained using the Wigner function. While the Wigner function of a coherent state, being a 2D Gaussian, can be readily interpreted in terms of the number and the phase uncertainty of this state the same cannot be said for the Fock states. The radial variable $r = \sqrt{x^2+p^2}$ of the Wigner function $W(x,p)$ for Fock states has a Laguerre polynomial dependence and thus it is not sharply defined in marked contrast to the Kronecker delta dependence of the "radial" variable n of $S_{NP}(n,\theta)$ in (15). The reason for this is that the square of the radial variable $r^2 = x^2 + p^2$ of $W(x,p)$ is associated with the *symmetrical ordering* of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ with respect to $\hat{a}$ and $\hat{a}^\dagger$ and not $\hat{N}$ itself [37]. That is,

$$\overline{r^{2m}} = \int W(x,p)(x^2+p^2)^m dxdp = \langle[(\hat{a}^\dagger\hat{a})^m]_{\rm sym}\rangle \tag{17}$$

where $[...]_{\rm sym}$ represents the symmetrical ordering [37] with respect to $\hat{a}$, $\hat{a}^\dagger$ and we find $\overline{r^2} = \langle\hat{N}+\frac{1}{2}\rangle$, $\overline{r^4} = \langle\hat{N}^2+\hat{N}+\frac{1}{2}\rangle$, $\overline{r^6} = \langle\hat{N}^3+\frac{3}{2}\hat{N}^2+2\hat{N}+\frac{3}{4}\rangle$, etc. The square of the radial variable is clearly not associated directly with the number operator. Moreover, while the angular variable $\phi = \arctan(p/x)$ does have phase-like properties [38], nevertheless, it is associated with the distribution

$$\tilde{W}(\phi) = \int_0^\infty W(r\cos\phi, r\sin\phi)dr$$

which can have negative values and which can differ significantly from the (non-negative) canonical phase distribution $P(\theta)$ given by (13) for some states [39]. Thus the angular variable of the Wigner function $W(x,p)$ does not represent the phase observable. This places $S_{NP}(n,\theta)$ in the unique position as being the only quasiprobability distribution currently available that gives a *direct representation of number and phase properties and thus of number-phase complementarity*.

Next we consider the state given by the superposition of two binomial states

$$|\psi\rangle = C[|p,M\rangle + \exp(i\hat{N}\pi)|p,M\rangle]$$

where C is a normalization constant. In Fig. 2 we give a plot of the associated $S_{NP}(n,\theta)$ function for mean $\bar{n} = pM \approx 10$ for different values of p. For p very near 1 (e.g., $p = 0.99$

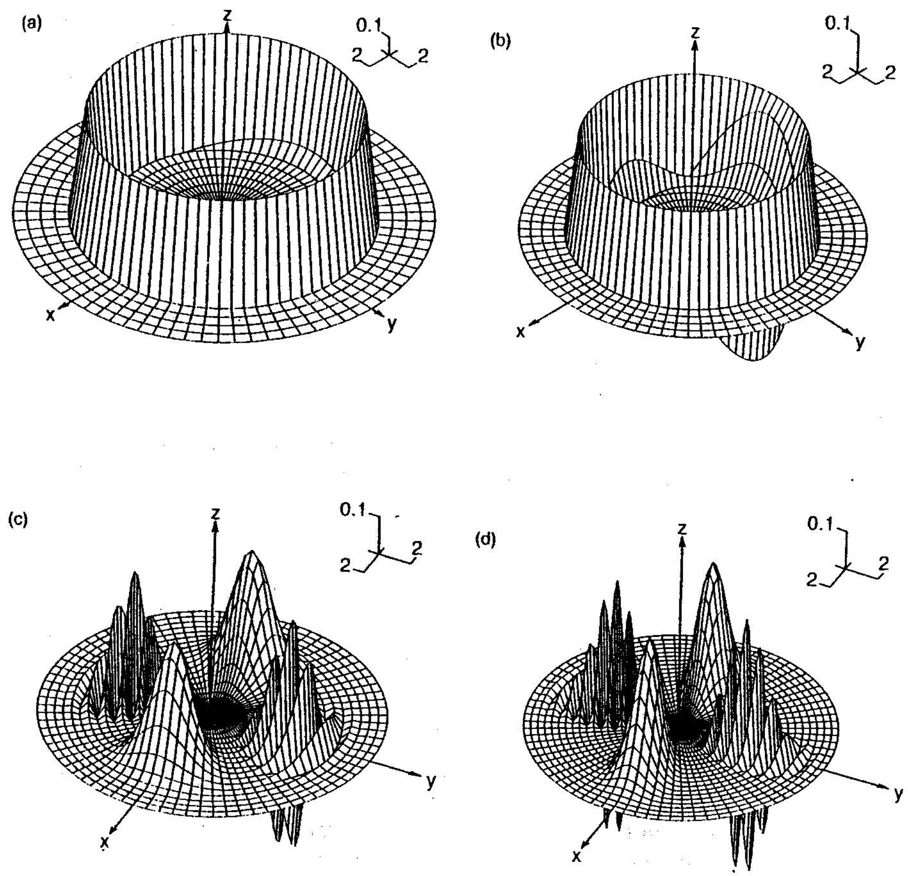

Fig. 2. The $S_{NP}(n,\theta)$ representation of the superposition of binomial states $C[|p,M\rangle + \exp(i\hat{N}\pi)|p,M\rangle]$. The values of M and p are the same as in Fig. (1). Interference fringes typical of Schrödinger cat states emerge as the component states become distinguishable and approach coherent states.

in Fig. 2 (a)) we find the $S_{NP}(n,\theta)$ function approaches that of the Fock state $|10\rangle$. As we reduce the value of p the features of $S_{NP}(n,\theta)$ take a drastic change. We note that the modulus of the overlap of the components of the binomial superposition $|\psi\rangle$ is given by $|\langle p,M|\exp(i\hat{N}\pi)|p,M\rangle| = |1-2p^2|^{\bar{n}/p^2}$ which is approximated by $\exp[-2\bar{n}(1-p^2)/p^2]$ for $p^2 \approx 1$. The e^{-1} point for $\bar{n} = 10$ occurs at $p \approx 0.97$ and the overlap becomes very small (≈ 0.002) at $p = 0.9$. Thus the binomial superposition state comprises two distinguishable states at $p = 0.9$ and we expect to see the characteristic interference fringes associated with the Schrödinger cat states at this value of p. This is indeed the case as shown in Fig. 2 (b). In the figure there are two symmetrical ridges above the radial lines $\theta = 0$ and $\theta = \pi$ and some interference fringes above $\theta = -\pi/2$ and $\theta = \pi/2$. The almost constant height of the ring above the circle $n = 10$ can be thought of as resulting from constructive interference. The interference fringes become more and more prominent as we further reduce p as shown in Fig. 2 (c) where $p = 0.5$. At the same time two separate hills, which are quite similar to those representing coherent states, emerge above the radial lines $\theta = 0$ and $\theta = \pi$. We eventually obtain the $S_{NP}(n,\theta)$ representation of the even coherent state which is proportional to $(|\alpha\rangle + |-\alpha\rangle)$ with $\alpha = \sqrt{10}$ in the limit $p \to 0$ and $M \to \infty$ as illustrated in Fig. 2 (d). This analysis shows that the superposition of two binomial states allows a systematic study of the growth of the quantum interference in a Schrödinger cat state.

Negative binomial state

The negative binomial states defined by equation (3) are, contrary to the binomial states, an infinite superposition of the Fock states. The $S_{NP}(n,\theta)$ can be studied in the same way as for the binomial state. The $S_{NP}(n,\theta)$ representation of the negative binomial state, given by

$$S_{NP}(n,\theta) = (2\pi)^{-1}\left\{\sum_{k=-n}^{n} \frac{\exp(2ik\theta)q^n(1-q)^{w+1}(n+w)!}{w![(n+k)!(n-k)!]^{1/2}} + \sum_{k=-n}^{n-1} \frac{\exp(i(2k+1)\theta)q^{n-1/2}(1-q)^{w+1}(n+w)!}{w![(n+k)!(n-k-1)!]^{1/2}}\right\}, \quad (18)$$

is plotted in Fig. 3 for several negative binomial states with fixed mean photon number $\bar{n} = 10$ and varying w ($w = 40, 5, 2, 0$). For w large (Fig. 3 (a), $w = 40$) the $S_{NP}(n,\theta)$ of negative binomial state $|q,w\rangle$ closely resembles to that of a coherent state with $\bar{n} = 10$ to which it approaches in the limit $w \to \infty$ with $q = \frac{\bar{n}}{\bar{n}+1+w}$. For lower values of w the negative binomial states becomes increasingly super-Poissonian and correspondingly the $S_{NP}(n,\theta)$ representation departs significantly from that of a coherent state as shown in Figs. 3 (b), (c) and (d). The complementary nature of the photon number and phase observable is directly evident in these figures: as w reduces the number distribution broadens and the phase distribution narrows. This complementarity behaviour will be explored further in the next section.

In the limit $w \to 0$ the negative binomial state $|q,w\rangle$ becomes a quasi-thermal state with a photon number distribution $P_n = q^n(1-q)$ equal to that of a thermal state of the same mean photon number $\bar{n} = \frac{q}{(1-q)}$. It is interesting to note that this distribution maximizes the photon number entropy for fixed $\bar{n}$ and it is shared by many states ranging from mixed thermal states $\sum_n P_n|n\rangle\langle n|$ to pure states $\sum_{n=0}^{\infty}\sqrt{P_n}\exp(i\phi_n)|n\rangle$ with arbitrary real ϕ_n. Of these states, those with the *minimum* phase uncertainty (as measured, for example, by the phase variance) are given by

$$\sum_{n=0}^{\infty}\sqrt{P_n}\exp^{in\varphi}|n\rangle = e^{i\hat{N}\varphi}|q, w = 0\rangle$$

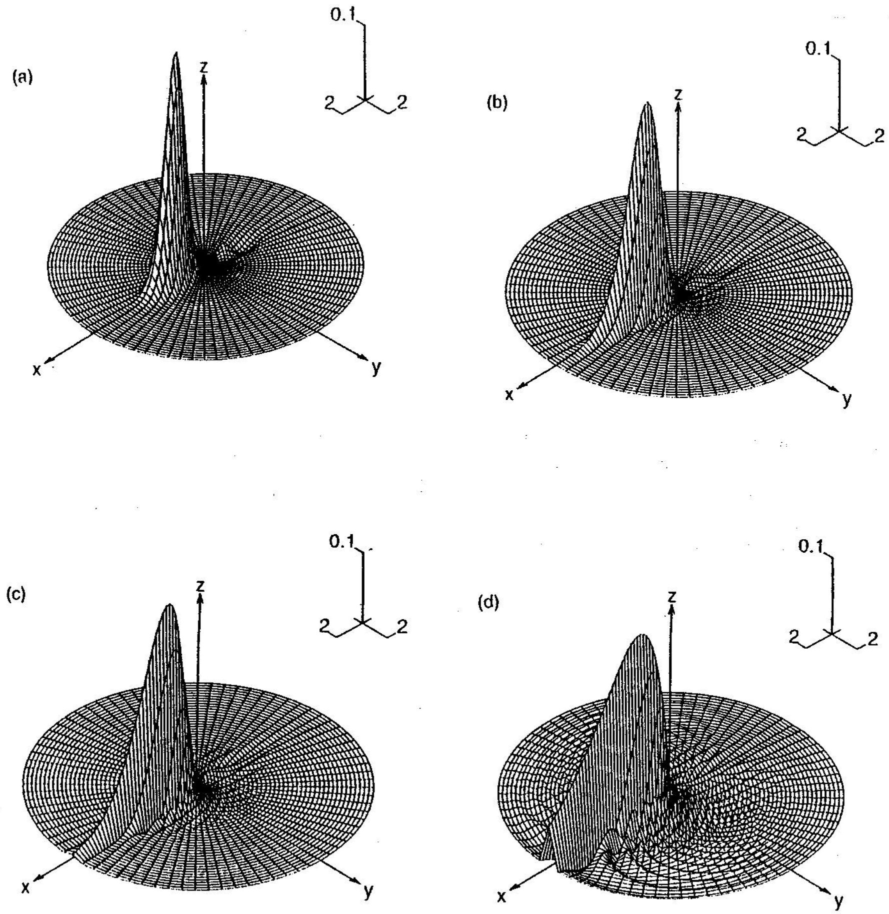

Fig. 3. The $S_{NP}(n,\theta)$ representation of negative binomial states $|q,w\rangle$ for fixed $\bar{n} = 10$. The values of w are (a) 40, (b) 5, (c) 2 and (d) 0. The representation evolves from one which is very similar to that of a coherent state to one which has more sharply-defined phase and a broader photon number.

for arbitrary real φ up to an unphysical overall phase factor. That is, phase-shifted negative binomial states are the maximum-photon-number-entropy states which have the minimum phase uncertainty. Fig. 3 (d) gives the $S_{NP}(n,\theta)$ representation of these extreme states. There is a relatively narrow ridge extending along $\theta = 0$ which illustrates the relatively sharply-defined phase of these states. The radial dependence of the height of the ridge is not the exponential $P_n = q^n(1-q)$ one might expect. Evidently the oscillations in $S_{NP}(n,\theta)$ in the plane surrounding the ridge contribute in the integral $P_n = \int S_{NP}(n,\theta)d\theta$ to yield the required exponential distribution P_n.

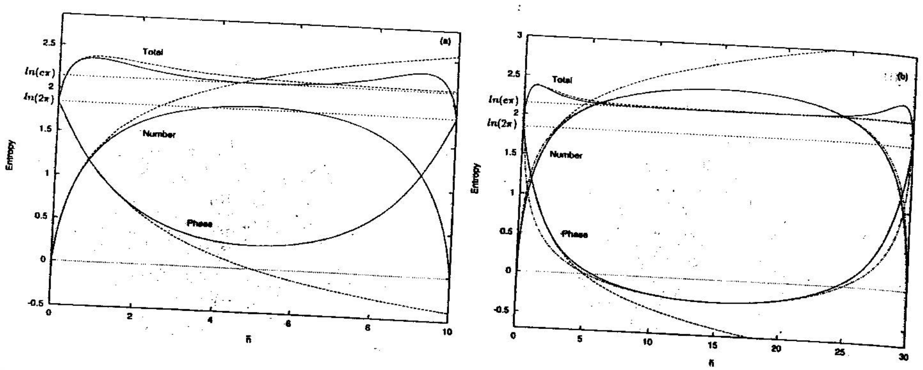

Fig. 4. The number and phase entropies for binomial states (solid curves) for (a) $M = 10$ and (b) $M = 30$ are plotted as a function of the mean photon number $\bar{n}$. Also shown are the corresponding entropies for coherent states (dashed curves) and the discrete and continuous limits, $\ln(2\pi)$ and $\ln(e\pi)$ (dotted lines). In (b) the approximate analytical expressions for the entropies Eqs. (28) and (29) for binomial states are plotted as dash-dotted curves.

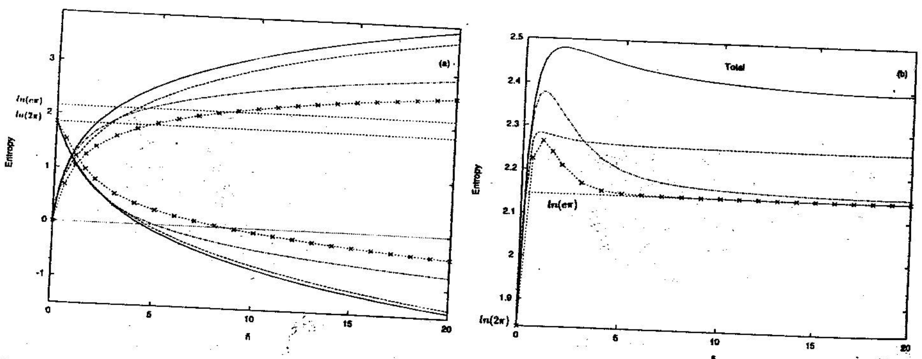

Fig. 5. Entropic uncertainties for various states for binomial (dotted curves with crosses), coherent (dashed-dotted curves), Airy (dashed curves) and negative binomial (solid curves) states. Negative binomial states clearly have the least phase entropy of all the states examined including Airy states.

4. Entropic uncertainties for number and phase

The entropic uncertainty relation for number and phase ([23], see also [25])

$$R_\phi + R_N \geq \ln(2\pi) \tag{19}$$

gives the lower bound of the sum of the Shannon entropics R_ϕ, R_N associated with the phase and photon number probability distributions $P(\theta) = |\langle\theta|f\rangle|^2$, $P_n = |\langle n|f\rangle|^2$ for

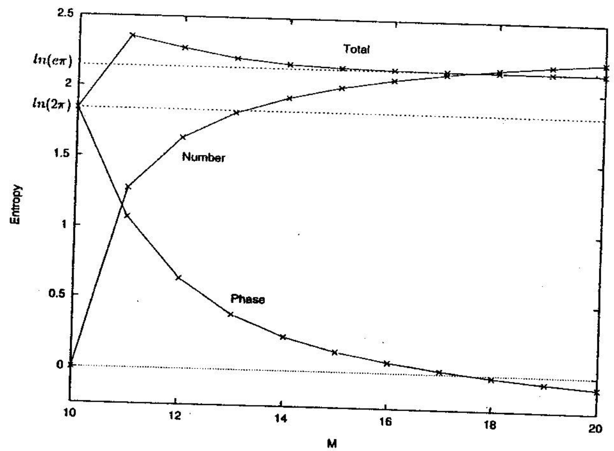

Fig. 6. Entropic uncertainties (crosses) for binomial states for fixed $\bar{n} = 10$ as a function of the integer variable M.

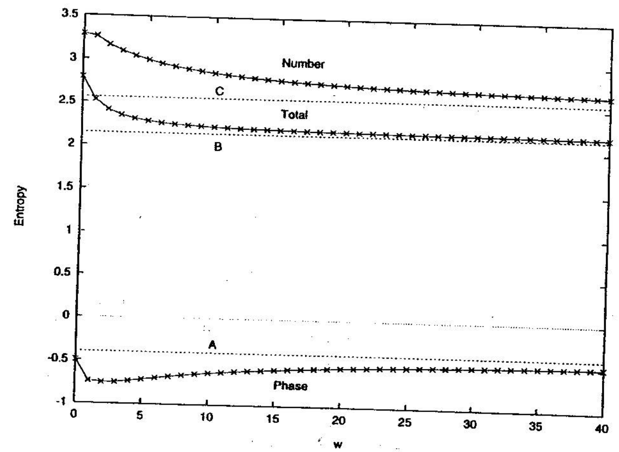

Fig. 7. Entropic uncertainties (crosses) for negative binomial states for fixed $\bar{n} = 10$ as a function of the integer variable w. The phase entropy is minimized at $w = 2$. Also shown for comparison are the corresponding entropies for a coherent state of the same mean photon number (dotted lines A, B, C)

any given state $|f\rangle$ where

$$R_\phi = -\int_{2\pi} P(\theta)\ln P(\theta)d\theta, \tag{20}$$

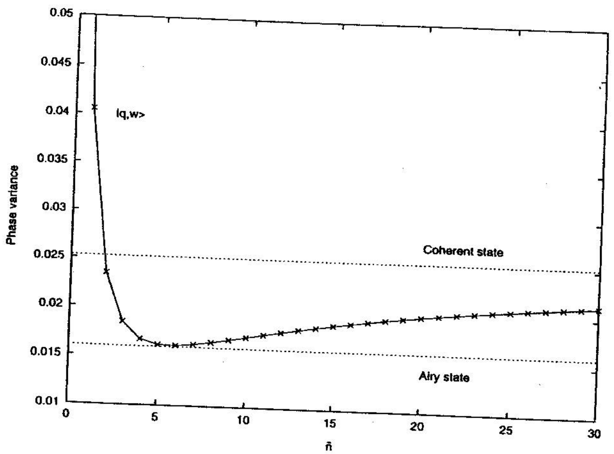

Fig. 8. Phase variance $\langle \Delta\hat{\phi}^2 \rangle$ (crosses) for negative binomial states for fixed $\bar{n} = 10$ as a function of the integer variable w. The phase variance is minimized at $w = 6$. Also shown are the phase variances for coherent and Airy states of the same mean photon number (dotted lines).

$$R_N = -\sum_{n=0}^{\infty} P_n \ln P_n . \tag{21}$$

The only physical states (i.e. states with finite moments of the photon number [5]) to satisfy the equality in (19) are the number states for which $R_N = 0$ and $R_\phi = \ln(2\pi)$. All other physical states give an entropic sum $R_\phi + R_N$ which is greater than $\ln(2\pi)$. In particular, the entropic sum for the coherent states $|\alpha\rangle$ with $\bar{n} = |\alpha|^2 \gg 1$ is approximately $\ln(e\pi)$ [23, 25]. This is an example of a more general result as follows. If the Fock state coefficients $\langle n|f\rangle$ of a given state $|f\rangle$ change relatively "smoothly" with increasing n such that (i) the sum in

$$\langle\theta|f\rangle = (2\pi)^{-1} \sum_{n=0}^{\infty} \exp(-in\theta)\langle n|f\rangle$$

can be approximated reasonably well by an integral, i.e.,

$$\langle\theta|f\rangle \approx \tilde{f}(\theta) \equiv \frac{1}{\sqrt{2\pi}} \int \exp(-ix\theta) f(x)dx \,, \tag{22}$$

and (ii) R_N can be approximated reasonably well with the sum in (21) replaced by an integral, i.e.,

$$R_N \approx -\int_0^{\infty} |f(x)|^2 \ln|f(x)|^2 dx \tag{23}$$

where $f(x) \in L^2$ is a suitable function of the continuous real variable x with the property that $f(n) \approx \langle n|f\rangle$ for non-negative integer n, then using the fact that the sum of the entropies associated with $|\tilde{f}(\theta)|^2$ and $|f(x)|^2$ is bounded below according to [20]

$$-\int |\tilde{f}(\theta)|^2 \ln|\tilde{f}(\theta)|^2 d\theta - \int |f(x)|^2 \ln|f(x)|^2 dx \geq \ln(e\pi) \tag{24}$$

we find

$$R_\phi + R_N \gtrsim \ln(e\pi). \tag{25}$$

The right-hand side of (25) is, in fact, the lower bound for the position-momentum entropic uncertainty relation [20]; its value of $\ln(e\pi)$ is significantly larger than the lower bound $\ln(2\pi)$ in (19). Thus (25) gives a more restrictive lower bound for states for which the approximations in (22) and (23) are valid.

For the particular case of an intense coherent state $|\alpha\rangle$ a suitable function $f(x)$ is given by the Gaussian $f(x) = (2\pi\bar{n})^{-1/4} \exp[-(x-\bar{n})^2/4\bar{n} + ix\phi]$ where $|\alpha|^2 = \bar{n} \gg 1$ and $\phi = \arg(\alpha)$. This function $f(x)$ and its Fourier transform $\tilde{f}(\theta)$ satisfy the equality in (24) which again shows that intense coherent states give the approximate equality in (25).

Both bounds in (19) and (25) will be important in our analysis below. We shall refer to them as the discrete ($\ln(2\pi)$) and the continuous ($\ln(e\pi)$) bounds since they are associated with discrete and continuous distributions, respectively.

Binomial states

The simplest, nontrivial binomial state $|p, M\rangle$ occurs for $M = 1$ in which case it is a superposition of just two Fock states:

$$|p, M = 1\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle. \tag{26}$$

The number and phase entropies for this two-state superposition have already been calculated by Rojas González *et al.* [25]. In particular, the entropic sum $R_\phi + R_N$ was found to rise smoothly from from the discrete bound of $\ln(2\pi) \approx 1.84$ for $p = 0$ to a maximum of $\ln(8\pi/e) \approx 2.22$ for an equally-weighted superposition at $p = 1/2$ and back to $\ln(2\pi)$ at $p = 1$. Clearly the increase in number entropy exceeds the decrease in phase entropy as the state $|p, M = 1\rangle$ changes from a single Fock state ($p = 0$) to an equal-weighted superposition of two Fock states ($p = 1/2$).

Similar behaviour is also evident in Figs. 4 (a) and 4 (b) where we have plotted the entropies R_ϕ, R_N and their sum (solid curves) for the binomial states $|p, M\rangle$ with $M = 10$ and $M = 30$ as a function of the mean photon number $\bar{n} = pM$. The curves representing the entropic sum $R_\phi + R_N$ rise rapidly from the discrete limit of $\ln(2\pi)$ to maxima of approximately 2.35 ($M = 10$) and 2.37 ($M = 30$) at $\bar{n} \approx 1$. In this region of low $\bar{n}$ the binomial states comprise of a superposition of just a few Fock states. For example, the first few Fock state coefficients $\langle n|p, M\rangle$ of the binomial state with $p = 0.1$ and $M = 10$ (i.e. $\bar{n} = 1$) are 0.59, 0.62, 0.44, 0.24 and 0.11 for $n = 0,1,2,3$ and 4 respectively. The rapid rise in $R_\phi + R_N$ between $\bar{n} = 0$ and $\bar{n} \approx 1$ can be attributed, therefore, to the increase in R_N exceeding the decrease in R_ϕ as the binomial state changes from being a single Fock state ($\bar{n} = 0$) to a superposition of just a few Fock states ($\bar{n} \approx 1$). This behaviour is also reminiscent of weak coherent states as found by Rojas González *et. al.* [25]. In fact binomial states $|p, M\rangle$ are closely approximated by coherent states of the same mean photon number $\bar{n}$ even for finite M provided $p = \bar{n}/M \ll 1$. This is verified in Figs. 4 (a), (b) which show that the entropies R_ϕ, R_N for the binomial state (solid curves) follow closely the corresponding entropies of a coherent state (dashed curves) of the same intensity for $\bar{n}$ up to about 1.

As the value of $\bar{n}$ increases the entropic sum $R_\phi + R_N$ tends towards the continuous bound of $\ln(e\pi)$. This approach to the continuous bound is readily explained in recalling the De Moivre-Laplace approximation: for $p \sim 1/2$ and $M \geq 10$ the binomial coefficients are approximated reasonably well by a Gaussian as follows,

$$\frac{M!}{n!(n-M)!}p^n(1-p)^{M-n} \approx (2\pi\Delta n^2)^{-1/2}\exp[-(n-\bar{n})^2/2\Delta n^2] \qquad (27)$$

where $\Delta n^2 = \bar{n}(1-\bar{n}/M)$ and $\bar{n} = pM$. Using this approximation for the number state coefficients of a binomial state and then approximating sums over n by appropriate integrals over n in the calculations of R_ϕ and R_N yields

$$R_N \approx \frac{1}{2}\ln(2\pi e\Delta n^2) \approx R_N^{coh} + \frac{1}{2}\ln(1-\bar{n}/M) \qquad (28)$$

$$R_\phi \approx \frac{1}{2}\ln(\pi e/2\Delta n^2) \approx R_N^{coh} - \frac{1}{2}\ln(1-\bar{n}/M) \qquad (29)$$

where $R_N^{coh} \approx \frac{1}{2}\ln(2\pi e\bar{n})$ and $R_\phi^{coh} \approx \frac{1}{2}\ln(\pi e/2\bar{n})$ are the corresponding entropies of a coherent state of the same intensity [25]. The validity of these results is illustrated in Fig. 4 (b) where the right-hand sides of (28) and (29) are plotted as dash-dotted curves. From (28) and (29) we find that while the entropies R_N, R_ϕ for binomial states differ significantly from the corresponding coherent state values for $\bar{n} \sim M/2$, the entropic sum $R_\phi + R_N$ is approximately the same as that of the coherent states of the same intensity.

There is a reflection symmetry in the entropic uncertainties for the binomial states about the point $\bar{n} = M/2$ (i.e., for $p = 1/2$) due to the symmetric nature of the Fock state coefficients in $|p, M\rangle$ on interchanging p with $1-p$. This symmetry can be seen in the solid curves in Figs. 4 (a),(b). The point of reflection, $p = 1/2$, is the point where the binomial state has the broadest spread in photon number and, correspondingly, the narrowest spread in phase. The question that arises naturally here is: how sharp is the phase at this point? To answer this we compare in Fig.6 the entropic uncertainties for binomial states with $p = 1/2$ with that of coherent and Airy states as a function of the mean photon number. Airy states are defined by

$$|A\rangle = C\sum_{n=0}^{\infty} \mathrm{Ai}[a^{-1/3}(n+\epsilon) - 2.34]|n\rangle,$$

where $\mathrm{Ai}(x)$ is the Airy function [40], $\epsilon = 0.86$, C is a normalization constant and a is an adjustable parameter that determines the mean photon number [33]. These states approximate phase optimized states which are states with the minimum phase variance for a fixed mean photon number [33, 34]. In particular the phase variance of Airy states is proportional to $1/\bar{n}^2$ in contrast to $1/\bar{n}$ for coherent states. Fig. 5 (b) shows that binomial states are closer to minimum entropic uncertainty states compared to the coherent and Airy states. However, Fig. 5 (a) shows that both Airy and coherent states have lower phase entropy for the same mean photon number.

We turn now to a sequence of binomial states similar to that represented in Figs 1(a) to 1(d). In Fig. 6 we plot the number and phase entropies and their sum for the binomial state $|p, M\rangle$ with fixed $\bar{n} = 10$ as a function of M. The curves clearly display the transition of the number and phase properties of the states as they interpolate between the Fock state $|10\rangle$ at $M = 10$ to approximately the coherent state $|\alpha\rangle$ with $|\alpha| = \sqrt{10}$ for $M \approx 20$. The maximum in $R_\phi + R_N$ at $M = 11$ can also be attributed to the binomial state at this point being a superposition of a few Fock states similarly to the corresponding maxima in Figs. 4 (a), (b).

Negative binomial states

In Fig. 7 we have plotted the entropies R_N, R_ϕ and their sum for the negative binomial state $|q, w\rangle$ for fixed $\bar{n} = \frac{(1+w)q}{(1-q)} = 10$ as a function of w. This sequence of states follows the transition from a quasi-thermal state ($w = 0$) to an approximate coherent state ($w \geq 40$) as depicted by the $S_{NP}(n, \theta)$ representation in Fig. 3 (a)-(d). The photon statistics of these states correspondingly changes from super-Poissonian to Poissonian and we find, not unexpectedly, that the number entropy in Fig. 7 decreases with increasing w. Complementarity suggests that the phase distribution should correspondingly broaden and thus the phase entropy should increase. This is indeed the case for $w > 2$. But what is perhaps surprising is that the phase entropy increases as w approaches 0 from $w = 2$. The reason for this [32] can be traced to the fact that although the width of the peak in the phase probability distribution $P(\theta)$ reduces with decreasing w, the peak in $P(\theta)$ "sits" on a broad plateau whose height is largest for $w = 0$. Clearly there is a non-zero value of w which gives a lower phase entropy than that of both coherent states and quasi-thermal states of the same intensity. For $\bar{n} = 10$ the optimum value of w is 2. We have found by numerical analysis that the optimum value of w depends on the value of $\bar{n}$. For example $w = 2$ gives the minimum phase entropy for values of $\bar{n}$ from 6.1 to 36.4, whereas $w = 3$ gives the minimum for $\bar{n}$ from 3.8 to 6.1.

It is interesting to compare the behaviour of the phase entropy with that of the phase variance found by Gantsog *et. al.* [32] recently. In Fig. 8 we have plotted the phase variance [5, 10]

$$\langle\Delta\hat{\phi}^2\rangle = \langle\hat{\phi}^2\rangle - \langle\hat{\phi}\rangle^2 \qquad (30)$$

where

$$\langle\hat{\phi}^n\rangle = \int_{-\pi}^{\pi}\theta^n P(\theta)d\theta \qquad (31)$$

of the same sequence of negative binomial states as in Fig. 7. Also shown in this figure are the phase variance of a coherent and Airy state of the same intensity (dotted lines).

We immediately see that the phase variance is, for the most part, less than that of a coherent state and, in fact, approaches that of an Airy state at $w = 6$. Moreover, the point where the phase variance is minimized ($w = 6$) differs markedly from the point where the phase entropy is minimized ($w = 2$). This difference can be explained by noting that entropy depends primarily on the "height" of a distribution whereas the variance depends on the "distance" from the mean with the net result being that the phase variance is more sensitive to the rising plateau in $P(\theta)$ as w approaches 0 than the phase entropy.

Finally we compare the number and phase entropies of the negative binomial states with that of the other states considered in this paper. The solid curves in Fig. 5 (a), (b) are the entropies R_N, R_ϕ and their sum for the negative binomial state $|q, w\rangle$ for which the value of w is chosen to minimize the phase entropy. The important point here is that the phase entropy of negative binomial states is lower than that of the all other states considered including the Airy states. This clearly shows that while Airy states give approximately the minimum phase variance, they do not give the minimum phase entropy for a given intensity. What states do give minimum phase entropy remains an open question which we are currently investigating.

5. Discussion

We have examined the number-phase complementarity displayed by binomial and negative binomial states, $|p, M\rangle$ and $|q, w\rangle$, as they interpolate between Fock and coherent states and coherent and quasi-thermal states, respectively. We illustrated the utility of the number-phase Wigner function $S_{NP}(n,\theta)$ to give a direct graphical representation of this complementarity. We highlighted the differences between Wigner's quasiprobability function $W(x,p)$ and $S_{NP}(n,\theta)$. In particular, while Wigner's quasiprobability function $W(x,p)$ is very useful for representing the quantum nature of position and momentum it does not give, in contradistinction to $S_{NP}(n,\theta)$, a direct representation of photon number and phase, since e.g., the marginals of $W(x,p)$ are not the number and phase distributions. This shows the advantages of using $S_{NP}(n,\theta)$ for studies in which the number and phase properties of states are of interest.

We also examined the number-phase entropic uncertainty relations for the binomial and negative binomial states and found approximate expressions for the photon number and phase entropies for binomial states. We found that the photon number and phase entropies, R_N and R_ϕ, respectively, for relatively weak binomial states with $p = \bar{n}/M \ll 1$ are approximately the same as that of coherent states of the same mean photon number $\bar{n}$. While the sum $R_N + R_\phi$ for the binomial states remains approximately the same as that for coherent states as the intensity of the states is increased, however, the phase entropy of a binomial state is greater than that of the corresponding coherent state.

For negative binomial states we examined the values of w that minimize the phase entropy and found that for the range of $\bar{n}$ values studied the optimum values of w were finite and nonzero. That is, the optimum states are intermediate between quasi-thermal states ($w = 0$) and coherent states ($w \to \infty$). We also found that the phase variance of negative binomial states with $\bar{n} = 10$ closely approaches (from above) that of an Airy state of the same intensity. This extends recent work by Gantsog *et al.* [32]. The important point here is that Airy states give approximately the least phase variance for a given $\bar{n}$ [33, 34] and thus it indicates that negative binomial states can have extremely low phase uncertainty. Indeed, we found that negative binomial states give a ***lower phase entropy*** than that of an Airy state of the same intensity. This last result underscores the importance of solving the yet-unsolved problem of finding the set of states that give the minimum phase entropy for a given intensity.

In conclusion, we have used the number-phase Wigner function and the number-phase entropic uncertainty relation to gain further insight into the nature of the photon number and phase properties of binomial and negative binomial states.

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