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TWO-DIMENSIONAL THERMAL RADIATION IN DISK-LIKE RESONATORS[1]

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In disk-like resonators where the size in one direction is small in comparison to the sizes in the two perpendicular directions, the range of low frequencies corresponds to a two-dimensional mode structure with a two-dimensional mode density that is explicitly discussed for a rectangular resonator. This leads to a two-dimensional black-body radiation law if the temperature correlated with resonator size are small enough to occupy only these modes according to Bose-Einstein statistics and if there is a lot of two-dimensional modes within the new radiation contour to justify the two-dimensional continuum approximation. We discuss the conditions for this case and the corresponding laws for energy and entropy and for the radiation pressure which is a negative pressure onto the plates. The Casimir effect which remains for the absolute temperature equal to zero and which has also a negative value is the residual effect from the difference of the zero-point energies of the resonator configuration in comparison to the free space and is shortly discussed.

1. Introduction

At the last instant of the last century (or the beginning of our century?) a new constant h was introduced into physics by Planck [1-3] to establish a new law for the thermal equilibrium radiation of a black body which is in good agreement with the observation and solved at once all difficulties with the known radiation laws at that time. This new constant rang in the age of quantum physics and found its place in the rigorous quantum theory which will accompany the physicists forever. A new derivation of Planck's formula by introduction of spontaneous and stimulated emission was given by Einstein [4,5]. Plancks radiation law uses the fundamental assumption that the

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statistics of the basic harmonic oscillators of the radiation field with the frequency ν is obtained as the statistics of the oscillator with discrete equally spaced energies in steps of $h\nu$. Another less fundamental substitution makes the radiation law independent of the resonator form and makes it to a universal law depending only on the temperature as a parameter and some universal constants and makes quantities as the total energy proportional to the resonator volume. This is the transition from the discrete distribution of the frequencies of the resonator modes to a continuous distribution. The last assumption is justified under "usual" conditions of not too low temperatures and "three-dimensional" resonators with not too large differences of the sizes in different directions. Under extreme deviations of the resonator form from a body with approximately equal sizes in all directions one has deviations of the black-body radiation law. We treat such a case where the resonator form is disk-like with large sizes in two transversal directions in comparison to the size in one longitudinal direction. Then one has a range of low frequencies with a mode density corresponding to a two-dimensional mode structure and if the temperature is low enough to excite essentially only these oscillators then one has a "two-dimensional" black-body radiation. We discuss the conditions under which this has to be expected. For zero absolute temperature there remains the Casimir effect (e.g., [6–12]) resulting from the changes of the zero-point energy of the resonator in comparison to the free space. This effect can be only treated by the differences of the somehow truncated (infinite) zero-point energies. The thermal effects of two-dimensional mode structures or of other deviations from the "universal" three-dimensional mode structure can be additively separated from the Casimir effect in a way that it makes not necessary to include the zero-point energy from the beginning into our treatment.

2. Gibbs statistics of one oscillator mode

Equilibrium thermodynamics is based on statistical theory and describes large systems by mean values of additive quantities such as energy and entropy in dependence on the temperature and proportional to the volume. The theory of fluctuations of these quantities is already the next step beyond the equilibrium thermodynamics. Therefore, equilibrium thermodynamics does not need the knowledge of the exact state of a macroscopic system and connects only a few fixed parameters which can be realized by ensembles of systems with microscopically different complete sets of parameters. In case of the free radiation field of the harmonic oscillators of a resonator with ideally reflecting walls the necessary information consists in the frequencies of the resonator modes its degeneracies and the mean photon numbers in each mode depending on the temperature.

The radiation oscillators do not interact directly with each other but interact only through the walls by absorption and emission processes. This causes that the mean value of the total number of photons in the resonator is not a given fixed quantity which has to be distributed between the different modes but depends on the temperature. The maximally available information for one oscillator is contained in the density operator for this oscillator which in thermal equilibrium with temperature T ($\hbar \equiv h/(2\pi)$, k

Boltzmann constant) is given by

$$\varrho = \frac{1}{Z_\omega}\sum_{n=0}^{\infty} \exp\left(-n\frac{\hbar\omega}{kT}\right)|n\rangle\langle n|, \quad Z_\omega \equiv \sum_{n=0}^{\infty}\exp\left(-n\frac{\hbar\omega}{kT}\right) = \frac{e^x}{e^x-1}, \quad x \equiv \frac{\hbar\omega}{kT}. \tag{1}$$

This is, in principle, Gibbs statistics (extended to its quantum-mechanical version) of the occupation of the energy levels of the harmonic oscillators being subsystems of a large system in thermal equilibrium and characterized by maximum of entropy (e.g., [13]). As the entropy with the property of additivity for systems consisting of independent subsystems has to be taken the von Neumann entropy which is the expectation value of $S \equiv -\log\varrho$, i.e. $\overline{S} = -\langle\varrho\log\varrho\rangle$, ($\langle\ldots\rangle$ trace). As the "physical" entropy $\overline{S}$ compatible with statistical mechanics has to be taken $\overline{S} \equiv -k\langle\varrho\log\varrho\rangle$, ($k$ Boltzmann constant). The quantity Z_ω is the statistical sum for the oscillator mode.

For the mean number of photons $\overline{N_\omega}$ of an arbitrary mode with frequency ω and its dispersion $\overline{(\Delta N_\omega)^2}$ one obtains

$$\overline{N_\omega} = \frac{1}{e^x - 1}, \qquad \overline{(\Delta N_\omega)^2} = \frac{e^x}{(e^x-1)^2} = \overline{N_\omega}\left(1+\overline{N_\omega}\right). \tag{2}$$

The values $\overline{N_\omega}$ correspond to Bose-Einstein statistics with chemical potential equal to zero according to not fixed total number of photons. Orthogonal modes with the same frequency ω have to be taken into account according to the degree of degeneracy of the frequency. Furthermore, one finds for the entropy and for the entropy fluctuations of one oscillator mode of frequency ω

$$\overline{S_\omega} = k\left\{\log\left(1+\overline{N_\omega}\right) + \overline{N_\omega}\log\left(1+\frac{1}{\overline{N_\omega}}\right)\right\} = k\left\{\log\left(\frac{e^x}{e^x-1}\right) + \frac{x}{e^x-1}\right\},$$
$$\overline{(\Delta S_\omega)^2} = k^2\overline{N_\omega}\left(1+\overline{N_\omega}\right)\left(\log\left(1+\frac{1}{\overline{N_\omega}}\right)\right)^2 = k^2\frac{x^2e^x}{(e^x-1)^2}. \tag{3}$$

The mean values of the energy in one mode and its dispersion are

$$\overline{E_\omega} = \hbar\omega\overline{N_\omega} = kT\frac{x}{e^x-1}, \qquad \overline{(\Delta E_\omega)^2} = (\hbar\omega)^2\overline{(\Delta N_\omega)^2} = (kT)^2\frac{x^2e^x}{(e^x-1)^2}. \tag{4}$$

One finds

$$\frac{\partial\overline{S_\omega}}{\partial\overline{E_\omega}} = \frac{1}{T}, \qquad T = \frac{\hbar\omega}{k\log\left(1+\frac{1}{\overline{N_\omega}}\right)}, \tag{5}$$

independent of the mode and therefore of the frequency as it follows for the subsystems of oscillators from the maximum of entropy in the whole resonator in thermal equilibrium. If we look to $\overline{E_\omega}$ or $\overline{N_\omega}$ as the primarily given quantity for one mode then, under supposition of thermal equilibrium, (5) can be considered as the definition of the temperature.

3. Discrete distributions of mode frequencies contrary to universality of continuum approximation

An ideal resonator can be defined as a resonator with walls for which the permittivity becomes infinite for all frequencies (often called super-conducting medium). However, it is clear that this condition can be satisfied due to the unavoidable dispersion at least only for discrete frequencies (frequency zero in case of superconductors). Then the only boundary condition for such ideal resonators is that the tangential components of the electric field have to vanish at the walls. There are only a few resonator configurations, even in the ideal case of the boundary conditions, for which the possible resonator modes can be exactly calculated in an easy way. This is first of all the rectangular resonator with arbitrary side lengths. Furthermore, the circular-cylindrical resonator and the spherical resonator are relatively easy to treat. The spherical resonator has no parameters for a transition to disk-like resonators with one small size in comparison to two other sizes but it can serve as a good illustration for the universality of the continuous approximation of the mode density in the three-dimensional case. We consider here only in short form the rectangular resonator and make the transition to the two-dimensional case.

We choose the coordinates (x, y, z) of the rectangular resonator in direction of the edges with the lengths (a_x, a_y, a_z) ordered according to $a_x \geq a_y \geq a_z$ in such a way that the resonator fills the volume $0 \leq x \leq a_x,\ 0 \leq y \leq a_y,\ 0 \leq z \leq a_z$. The well-known solution of the wave equations for the components of the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ of the modes inside the rectangular resonators under the ideal boundary conditions are

$$\begin{aligned} E_x(x,y,z,t) &= A_x \cos k_x x \sin k_y y \sin k_z z\, \mathrm{e}^{-i\omega t} + c.c., \\ E_y(x,y,z,t) &= A_y \sin k_x x \cos k_y y \sin k_z z\, \mathrm{e}^{-i\omega t} + c.c., \\ E_z(x,y,z,t) &= A_z \sin k_x x \sin k_y y \cos k_z z\, \mathrm{e}^{-i\omega t} + c.c.\,. \end{aligned} \tag{6}$$

From the Maxwell equations one finds for the components of the magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$

$$\begin{aligned} B_x(x,y,z,t) &= -i\frac{c}{\omega}\,(k_y A_z - k_z A_y) \sin k_x x \cos k_y y \cos k_z z\, \mathrm{e}^{-i\omega t} + c.c., \\ B_y(x,y,z,t) &= -i\frac{c}{\omega}\,(k_z A_x - k_x A_z) \cos k_x x \sin k_y y \cos k_z z\, \mathrm{e}^{-i\omega t} + c.c., \\ B_z(x,y,z,t) &= -i\frac{c}{\omega}\,(k_x A_y - k_y A_x) \cos k_x x \cos k_y y \sin k_z z\, \mathrm{e}^{-i\omega t} + c.c.\,. \end{aligned} \tag{7}$$

The quantities (k_x, k_y, k_z) and ω are connected and restricted by

$$\begin{aligned} &k_x = n_x\frac{\pi}{a_x}, \quad k_y = n_y\frac{\pi}{a_y}, \quad k_z = n_z\frac{\pi}{a_z}, \qquad \omega = c\sqrt{k_x^2 + k_y^2 + k_z^2}, \\ &n_x, n_y, n_z = 0, 1, 2, \ldots, \qquad n_x n_y + n_y n_z + n_z n_x \geq 1. \end{aligned} \tag{8}$$

Behind this is the coupling of 8 plane monochromatic waves with the 8 possible combinations of wave vectors $\boldsymbol{k} = (\pm k_x, \pm k_y, \pm k_z)$ which contribute by superposition to each mode. The last restriction $n_x n_y + n_y n_z + n_z n_x \geq 1$ is made because in case that two of the three numbers (n_x, n_y, n_z) vanish the electric and magnetic field and thus the whole solution becomes vanishing. From the three complex amplitudes (A_x, A_y, A_z) are maximum only two linearly independent because from $\boldsymbol{\nabla E}(\boldsymbol{r}, t) = 0$ it follows

$$k_x A_x + k_y A_y + k_z A_z = 0. \tag{9}$$

In case that all three wave numbers (k_x, k_y, k_z) are nonvanishing one has 2 linearly independent solutions and one can choose, however, in a manifold of different ways two orthogonal waves in the sense that the energy or the Hamiltonian is the sum of two independent contributions and does not contain crossing terms of the chosen waves.

We now consider the special case that one of the three wave numbers (k_x, k_y, k_z) is equal to zero and assume for definiteness $k_z = 0$. One finds from (9–12) that the only remaining solution in this case corresponds to an amplitude A_z as follows

$$\begin{aligned} &E_x(x,y,z,t) = 0, \quad E_y(x,y,z,t) = 0, \quad B_z(x,y,z,t) = 0, \\ &E_z(x,y,z,t) = A_z \sin k_x x \sin k_y y\, \mathrm{e}^{-i\omega t} + c.c., \\ &B_x(x,y,z,t) = -i\frac{c}{\omega} k_y A_z \sin k_x x \cos k_y y\, \mathrm{e}^{-i\omega t} + c.c., \\ &B_y(x,y,z,t) = +i\frac{c}{\omega} k_x A_z \cos k_x x \sin k_y y\, \mathrm{e}^{-i\omega t} + c.c., \quad (\,k_z = 0\,). \end{aligned} \tag{10}$$

The twofold degeneracy of the solutions in case of $k_x \neq 0,\ k_y \neq 0,\ k_z \neq 0$ is here reduced to nondegeneracy. It is important for our further considerations that both the Poynting vector $\boldsymbol{S} = (c/(4\pi))[\boldsymbol{E}, \boldsymbol{B}]$ and the momentum density $\boldsymbol{g} = (1/(4\pi c))[\boldsymbol{E}, \boldsymbol{B}]$ of the field in such resonator modes possess only components in x- and y-direction but do not possess a component in z-direction.

We now consider the mode densities in the general three-dimensional case and in the two-dimensional case (13). First we write ω in dependence on (n_x, n_y, n_z) and (a_x, a_y, a_z) and separate the volume $V \equiv a_x a_y a_z$ of the resonator as follows $(a_i \equiv \sqrt[3]{V}\alpha_i)$

$$\omega = \pi c\sqrt{\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2} + \frac{n_z^2}{a_z^2}} = \frac{\pi c}{\sqrt[3]{V}}\sqrt{\frac{n_x^2}{\alpha_x^2} + \frac{n_y^2}{\alpha_y^2} + \frac{n_z^2}{\alpha_z^2}}, \qquad \alpha_x\alpha_y\alpha_z = 1. \tag{11}$$

The normalized numbers $(\alpha_x, \alpha_y, \alpha_z)$ characterize the relative sizes of the rectangular resonator. If one makes a plot of $\omega\sqrt[3]{V}/(\pi c)$ in the order of their increase and taken with their degeneracies and count on the x-axis the number of modes up to the corresponding frequency then one gets a monotonously increasing curve $f(n)$ which is not very different for different resonators and approximates for increasing total number of considered modes to a universal function. This is shown in Fig. 1 for a spherical resonator. We see how on the background of the universal function $f_3(n)$ remains a weak individuality of the considered resonator form. Let us calculate this universal function $f_3(n)$.

The differential number d^3n of modes in a frequency interval $d\omega$ is obtained in its continuous approximation from

$$d^3n = 2dn_x \wedge dn_y \wedge dn_z = 2\frac{V}{\pi^3} dk_x \wedge dk_y \wedge dk_z = \frac{2V}{8\pi^3} 4\pi|k|^2 d|k| = \frac{V}{\pi^2 c^3}\omega^2 d\omega. \tag{12}$$

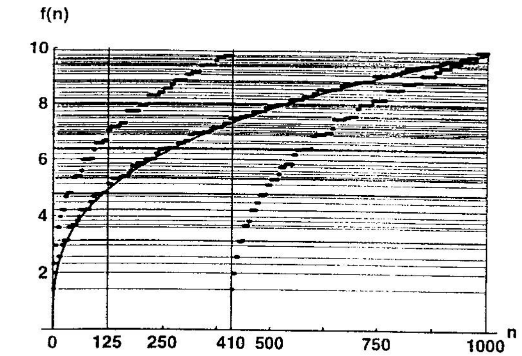

Fig. 1. Normalized frequencies of the first 1000 modes of an ideal spherical resonator counted with their degeneracies. The universal three-dimensional mode function $f_3(n)$ smelts together with the real curve in an excellent way. On the left up to $n = 410$ is the separate counting of the electric type and to the right of the magnetic type of modes.

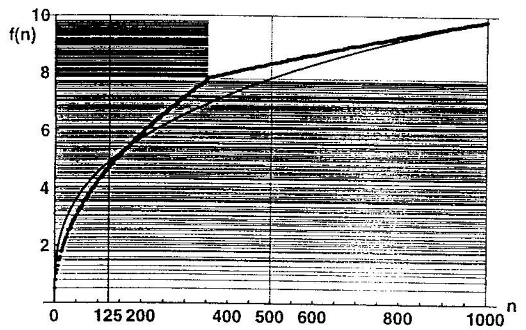

Fig. 2. Normalized frequencies of the first 1000 modes of an ideal rectangular resonator with $\alpha_x = \alpha_y = 2.8, \alpha_z = 1/7.84$ counted with their degeneracies. The universal three-dimensional mode function $f_3(n)$ shows significant deviations from the real curve which is here up to $f(n) \approx 7.84$ essentially the two-dimensional mode function $f_2(n)$ drawn in the normalized three-dimensional scheme. The transition to a mixed two-dimensional and three-dimensional mode function for $f(n) \geq 7.84$ can be distinctly seen. The modes seem to be more dense as for the spherical resonator but mainly due to less degeneracies as their counting shows.

Here the usual twofold degeneracy of modes was taken into account and the factor 8 was added in the denominator under the transition to the volume element in the wave-vector space because (n_x, n_y, n_z) and (k_x, k_y, k_z) are primarily restricted to nonnegative values. From this one obtains the three-dimensional mode density $\nu_3(\omega)$ and by integration over ω the total number $n = n_3(\omega)$ of modes up to a frequency ω in the form

$$\nu_3(\omega) \equiv \frac{d^3 n}{d\omega} = \frac{V}{\pi^2 c^3}\omega^2, \qquad n = n_3(\omega) \equiv \int_0^{\omega} d\omega' \nu_3(\omega') = \frac{V}{3\pi^2 c^3}\omega^3. \tag{13}$$

From this one finds by inversion of the function $n = n_3(\omega)$

$$\omega = \omega_3(n) = \frac{\pi c}{\sqrt[3]{V}} f_3(n), \qquad f_3(n) = \sqrt[3]{\frac{3}{\pi} n}, \qquad \sqrt[3]{\frac{3}{\pi}} = 0.984745. \tag{14}$$

Since it goes with the third root of n one obtains for each resonator of not too "extreme" form that the lowest 1000 modes go up to an approximate frequency $\omega \approx 9.85\pi c/\sqrt[3]{V}$.

We now consider disk-like resonators that corresponds for the rectangular resonator to $a_x \geq a_y \gg a_z$. Since $1/a_z^2$ is large compared with $n_x^2/a_x^2 + n_y^2/a_y^2$ for not too high (n_x, n_y) the modes of the lowest frequencies of such a resonator are formed with $n_z = 0$ corresponding to $k_z = 0$. The frequencies of these modes can be calculated by $((a_x, a_y) \equiv \sqrt{A}(\beta_x \beta_y))$

$$\omega = \pi c \sqrt{\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2}} = \frac{\pi c}{\sqrt[3]{V}} \sqrt{\frac{n_x^2}{\alpha_x^2} + \frac{n_y^2}{\alpha_y^2}} = \frac{\pi c}{\sqrt{A}} \sqrt{\frac{n_x^2}{\beta_x^2} + \frac{n_y^2}{\beta_y^2}}, \qquad A \equiv a_x a_y, \quad \beta_x \beta_y = 1. \tag{15}$$

A universal function of increase of ω with the counting of the number of modes is here obtained by normalization with the area A of the resonator transverse to z in analogy to the explained three-dimensional case. In figures with the normalization by the volume V the two-dimensional case forms curves with different ascents. The differential number $d^2 n$ of modes in a frequency interval $d\omega$ is obtained in the two-dimensional case from

$$d^2 n = dn_x \wedge dn_y = \frac{A}{\pi^2} dk_x \wedge dk_y = \frac{A}{4\pi^2} 2\pi |k| d|k| = \frac{A}{2\pi c^2} \omega d\omega. \tag{16}$$

A certain correction to $d^2 n$ could be obtained if we subtract $-(dn_x + dn_y)$ corresponding to the nonexisting modes with $n_x = 0$ or $n_y = 0$. We do not take this into account at this stage because it makes the thermodynamical calculations unnecessary complicated. The two-dimensional mode density $\nu_2(\omega)$ and by its integration over ω the total number $n = n_2(\omega)$ of modes up to a frequency ω in the range of reality of the two-dimensional case is obtained from (16) in the form

$$\nu_2(\omega) \equiv \frac{d^2 n}{d\omega} = \frac{A}{2\pi c^2}\omega, \qquad n = n_2(\omega) \equiv \int_0^{\omega} d\omega' \nu_2(\omega') = \frac{A}{4\pi c^2}\omega^2. \tag{17}$$

The inversion of the function $n = n_2(\omega)$ provides

$$\omega = \omega_2(n) = \frac{\pi c}{\sqrt{A}} f_2(n) = \frac{\pi c}{\sqrt[3]{V}} \sqrt[3]{\frac{a_z}{\sqrt{A}}} f_2(n), \qquad f_2(n) = \sqrt{\frac{4}{\pi} n}, \qquad \sqrt{\frac{4}{\pi}} = 1.12838\,. \tag{18}$$

This gives the two-dimensional universal function $\sqrt{A}\omega/(\pi c) = \sqrt{4n/\pi} = f_2(n)$ to which all two-dimensional cases of counting of resonator modes according to their strict increase of frequencies approximates. As an example this is shown in Fig. 2 for $\alpha_x = \alpha_y = 2.8, \alpha_z = 1/7.84$ in the same way as in Fig. 1. We see here the deviations of the two-dimensional mode function $f_2(n)$ from the three-dimensional mode function $f_3(n)$.

We now discuss the conditions of reality of the two-dimensional mode continuum and calculate the thermodynamic characteristics of the radiation field in a resonator in the three-dimensional case and in the two-dimensional case of disk-like resonators.

4. Conditions of reality of the continuum approximation and of the two-dimensional case

The continuum approximation in thermodynamics of the radiation field means the substitution of sums over the discrete resonator frequencies of the form $\sum_\omega \overline{N_\omega} f(\omega)$ by integrals $\int_0^\infty d\omega \overline{N_\omega} \nu_3(\omega) f(\omega)$ in the three-dimensional and $\int_0^\infty d\omega \overline{N_\omega} \nu_2(\omega) f(\omega)$ in the two-dimensional case. Herein, $f(\omega)$ are given polynomial functions of ω for which one has to calculate the expectation value in thermal equilibrium. With $\overline{N_\omega}$ according to (2) and with the substitution $x \equiv (\hbar\omega)/(kT)$ the continuum approximation leads to the calculation of integrals over functions $x^n/(e^x - 1)$ with small values $n \geq 1$ or over functions $x^n e^x/(e^x-1)^2$ for the calculation of fluctuations. All such functions possess their only maximum in the vicinity of $x = 1$ and decrease rapidly for $x \gg 1$, say $x > 10$. Therefore, it is important for the reality of the continuum approximation that the resonator possesses already a lot of modes in the range $x \leq 1$. If we take the lowest frequencies in case of resonators with comparable sizes in all directions for $x \equiv (\hbar\omega)/(kT)$ then this means

$$\frac{\hbar c}{k} \frac{\pi}{T\sqrt[3]{V}} \ll 1, \qquad \frac{\hbar c}{k}\pi = 0.22899 \cdot \pi \text{ K} \cdot \text{cm} = 0.719393 \text{ K} \cdot \text{cm}. \tag{19}$$

Only for small products of temperature T with resonator sizes $\sqrt[3]{V}$ this condition of reality of the continuum approximation can be violated, e.g., $T = 1$ K and $\sqrt[3]{V} = 1$ cm comes into this transition domain.

In case of disk-like resonators the condition for the reality of the two-dimensional continuum approximation is

$$\frac{\hbar c}{k} \frac{\pi}{T\sqrt{A}} \ll 1 \ll \frac{\hbar c}{k} \frac{\pi}{T a_z}, \tag{20}$$

where on the right-hand side the smallest resonator frequency with $n_z \neq 0$ was taken. This frequency is the approximate beginning of the range where the three-dimensional mode density becomes important. For example, for $a_z = 0.1$ cm and $\sqrt{A} \geq 10$ cm temperatures of about 1 K are necessary for the reality of the two-dimensional mode density. The lowest resonator frequencies are in this example in the GHz-range. For $a_z = 10^{-3}$ cm and $\sqrt{A} \geq 1$ cm the two-dimensional thermal radiation becomes already significant for room temperatures. Therefore, this can play a role in micro- or optoelectronics.

5. Comparison of the three-dimensional with the two-dimensional thermal radiation

The following two auxiliary formulas are necessary for the calculation of the thermodynamical functions of the radiation and of fluctuations

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = \int_0^\infty dx\, x^n e^{-x} \sum_{k=0}^\infty e^{-kx} = n! \sum_{k=0}^\infty \frac{1}{(k+1)^{n+1}} = n!\zeta(n+1),$$

$$\int_0^\infty dx \frac{x^n e^x}{(e^x-1)^2} = \int_0^\infty dx\, x^n e^{-x} \sum_{k=0}^\infty (k+1) e^{-kx} = n! \sum_{k=0}^\infty \frac{1}{(k+1)^n} = n!\zeta(n),$$

$$\zeta(2) = \frac{\pi^2}{6} = 1.64493, \quad \zeta(3) = 1.20206, \quad \zeta(4) = \frac{\pi^4}{90} = 1.08232, \tag{21}$$

where $\zeta(z)$ denotes the Riemann zeta function.

In the three-dimensional case one obtains from (4) and (13) the following radiation law (Planck's radiation) and total energy E (we omit the bars in the more consequent notation $\overline{E}$ and do not insert $\pi^4/90$ for $\zeta(4)$) (Fig. 3)

$$d\overline{E_\omega} = \frac{V(kT)^4}{\pi^2(\hbar c)^3} \frac{x^3}{e^x - 1} dx, \qquad E = \frac{6\zeta(4)}{\pi^2} \frac{(kT)^4}{(\hbar c)^3} V, \qquad x \equiv \frac{\hbar\omega}{kT}. \tag{22}$$

In the two-dimensional case one obtains from (4) and (17) (Fig. 3)

$$d\overline{E_\omega} = \frac{A(kT)^3}{2\pi(\hbar c)^2} \frac{x^2}{e^x - 1} dx, \qquad E = \frac{\zeta(3)}{\pi} \frac{(kT)^3}{(\hbar c)^2} A, \qquad x \equiv \frac{\hbar\omega}{kT}. \tag{23}$$

The total energy is here only proportional to the third power of the temperature and proportional to the area A of the disk-like resonator. For the total entropy S (we omit again bars) one finds in the three-dimensional and in the two-dimensional case

$$S = k\frac{8\zeta(4)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 V, \qquad S = k\frac{3\zeta(3)}{2\pi} \left(\frac{kT}{\hbar c}\right)^2 A. \tag{24}$$

It is interesting that the thickness a_z of the disk-like resonator does not appear in the two-dimensional case but only the area A. For the fluctuations of the total energy and total entropy in both cases we find using (3) and (4)

$$\frac{\overline{(\Delta E)^2}}{(kT)^2} = \frac{\overline{(\Delta S)^2}}{k^2} = \frac{24\zeta(4)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 V, \qquad \frac{\overline{(\Delta E)^2}}{(kT)^2} = \frac{\overline{(\Delta S)^2}}{k^2} = \frac{3\zeta(3)}{\pi} \left(\frac{kT}{\hbar c}\right)^2 A. \tag{25}$$

The relative energy and entropy fluctuations decrease with increasing temperature and are proportional to the reciprocal square root of the volume V or area A, respectively.

If one expresses the total energy E as a function of S and V in the three-dimensional case one obtains the first thermodynamic potential $E(S, V)$. In the two-dimensional case one can express E as a function of S, A and a_z and obtains the first thermodynamic potential $E(S, A, a_z)$, where the separate dependence on A and a_z expresses an anisotropy in transversal and longitudinal directions of a disk-like resonator. One obtains

$$E = E(S,V) = \frac{3}{8}\sqrt[3]{\frac{\pi^2}{\zeta(4)}} \left(\frac{S}{k}\right)^{\frac{4}{3}} \frac{\hbar c}{V^{\frac{1}{3}}}, \qquad E = E(S, A, a_z) = \frac{2}{3}\sqrt{\frac{2\pi}{3\zeta(3)}} \left(\frac{S}{k}\right)^{\frac{3}{2}} \frac{\hbar c}{A^{\frac{1}{2}}}, \tag{26}$$

proving that $E(S, A, a_z)$ does not depend on a_z. The (radiation) pressure P can be obtained in the three-dimensional case by the relations [13]

$$P = -\left(\frac{\partial E}{\partial V}\right)_S = \frac{1}{8}\sqrt[3]{\frac{\pi^2}{\zeta(4)}} \left(\frac{S}{kV}\right)^{\frac{4}{3}} \hbar c = \frac{2\zeta(4)}{\pi^2} \frac{(kT)^4}{(\hbar c)^3}, \qquad T = \left(\frac{\partial E}{\partial S}\right)_V, \tag{27}$$

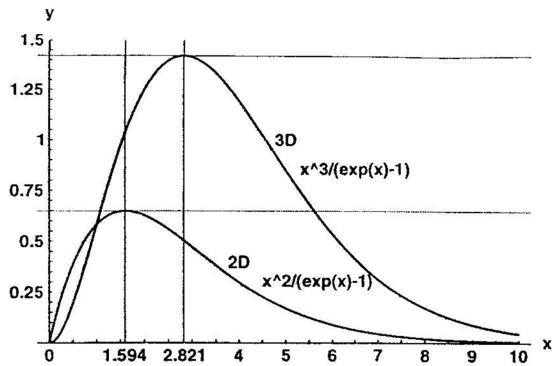

Fig. 3. Comparison of the three-dimensional and the two-dimensional thermal radiation law.

which are the state equations of the photon gas for adiabatic and isothermal changes. In the two-dimensional case one has to distinguish the pressure onto the resonator boundaries in transversal directions $P_\perp$ and in longitudinal direction (z-direction) P_z due to the anisotropy according to

$$P_\perp = -\frac{1}{a_z}\left(\frac{\partial E}{\partial A}\right)_{S,a_z} = \frac{1}{3}\sqrt{\frac{2\pi}{3\zeta(3)}}\left(\frac{S}{kA}\right)^{\frac{3}{2}}\frac{\hbar c}{a_z} = \frac{\zeta(3)}{2\pi}\frac{(kT)^3}{(\hbar c)^2}\frac{1}{a_z},$$
$$P_z = -\frac{1}{A}\left(\frac{\partial E}{\partial a_z}\right)_{S,A} = 0, \qquad T = \left(\frac{\partial E}{\partial S}\right)_{A,a_z}. \tag{28}$$

These state equations show that in the considered idealized disk-like resonator under conditions of the two-dimensional case there is no radiation pressure onto the basis walls and the pressure from outward according to (27) tries to press these walls to the inside. This is in agreement with the property of the momentum of the electromagnetic field which in the two-dimensional case does not possess a component in z-direction (see section 3). On the other hand, the pressure onto the side walls increases with the reciprocal value of a_z. In comparison of the three-dimensional and of the two-dimensional cases one has the relations

$$PV = \frac{E}{3}, \qquad P_\perp A a_z = PV = \frac{E}{2}. \tag{29}$$

The anisotropy can be also expressed by the macroscopic energy-momentum tensor of the macroscopic body [14] which becomes in the both cases

$$T_{\mu\nu} = \begin{pmatrix} \varepsilon, & 0, & 0, & 0 \\ 0, & P, & 0, & 0 \\ 0, & 0, & P, & 0 \\ 0, & 0, & 0, & P \end{pmatrix}, \qquad T_{\mu\nu} = \begin{pmatrix} \varepsilon, & 0, & 0, & 0 \\ 0, & P_\perp, & 0, & 0 \\ 0, & 0, & P_\perp, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}, \qquad \varepsilon \equiv \frac{E}{V} = \frac{E}{Aa_z}. \tag{30}$$

The trace $T_\nu^\nu = g^{\nu\mu}T_{\mu\nu}$ of the energy-momentum tensor is equal to 0 as it corresponds to a general property of this tensor for the free electromagnetic field.

The Helmholtz free energy F can be calculated either by the logarithm of the statistical sum Z multiplied by $-kT$ or via a Legendre transformation $F = E - TS$ of the total energy E regarding the variable S. One obtains in the three-dimensional case

$$F = F(T,V) = E - TS = -kT\log Z = -\frac{2\zeta(4)}{\pi^2}\frac{(kT)^4}{(\hbar c)^3}V = -\frac{E}{3}, \tag{31}$$

and in the two-dimensional case

$$F = F(T,A,a_z) = E - TS = -kT\log Z = -\frac{\zeta(3)}{2\pi}\frac{(kT)^3}{(\hbar c)^2}A = -\frac{E}{2}, \tag{32}$$

from which the radiation pressure can be obtained by

$$P = -\left(\frac{\partial F}{\partial V}\right)_T, \qquad P_\perp = -\frac{1}{a_z}\left(\frac{\partial F}{\partial A}\right)_{T,a_z}, \qquad P_z = -\frac{1}{A}\left(\frac{\partial F}{\partial a_z}\right)_{T,A}, \tag{33}$$

in agreement with (27) and (28). A more detailed discussion and comparison of the radiation laws for the three-dimensional and the two-dimensional case we delay to future.

6. Short remarks about the Casimir effect

The Casimir effect [6–12] is the residual effect from the change of the zero-point energy E_0 of the given resonator configuration in comparison to the free space and consists in case of parallel plates in an attractive force (negative pressure) onto the plates proportional to a_z^{-4} according to

$$P'_{0,z} \equiv P_{0,z} - P_0 = -\frac{\pi^2\hbar c}{240}\frac{1}{a_z^4}. \tag{34}$$

The derivation of this pressure by using a cutoff function of the frequency and by applying the Euler-Maclaurin summation formula [8,11] makes the impression that it does not critically depend on the cutoff function which is assumed to correspond to a cutoff at wavelength of the order of the atomic dimensions. Other calculations which we intend to publish rather indicate that it depends considerable on the cutoff frequencies and that the given result corresponds to cutoff frequencies of the order of $\pi c/a_z$ which is the frequency where the pure two-dimensional mode structure makes the transition into a mixed two-dimensional and three-dimensional mode structure. However, this is only a preliminary information which must be finally clarified. In comparison to the Casimir force, the residual pressure from the two-dimensional thermal radiation obtained from (28) and (29) has the (stable) form

$$P'_z \equiv P_z - P = -\frac{\pi^2\hbar c}{45}\left(\frac{kT}{\hbar c}\right)^4, \tag{35}$$

that means it is of the same structure as the Casimir force with $1/a_z$ substituted by $(kT)/(\hbar c)$ and has the same sign corresponding to an inward pressure but can be distinguished by its independence of distance a_z or dependence on temperature T.

7. Conclusion

It was shown that in disk-like resonators there is in dependence on the resonator sizes and on the temperature a range for which the thermodynamical equilibrium radiation is essentially a two-dimensional. This causes remarkable deviations from the known "three-dimensional" radiation laws but shows under discussed conditions a two-dimensional universal behaviour. The resonator modes of a rectangular resonator were considered under idealized boundary conditions although there are due to dispersion, principally, no media for which they can be satisfied for all frequencies. Nevertheless, to the usual belief, they provide reliable results because for equilibrium thermodynamics there exists a natural truncation of frequencies that makes it unnecessary to calculate the frequencies of the resonator modes very exactly in the range of high frequencies in comparison to the lowest resonator frequencies. Contrary to this, the treatment of the Casimir effect seems to be sensitive to the frequency cutoff. Thus the Casimir effect remains to be a very delicate effect with high challenges to the experiment.

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