acta physica slovaca vol. 47 No. 3/4, 327 – 330 June 1997

QUANTUM-STATE RECONSTRUCTION FOR DAMPED SYSTEMS[1]

T. Opatrný [†2], W. Vogel[‡], D.-G. Welsch[†]

[†] *Friedrich-Schiller-Universität Jena, Theoretisch-Physikalisches Institut, Max-Wien Platz 1, D–07743 Jena, Germany*

[‡] *Universität Rostock, Fachbereich Physik, Universitätsplatz 3, D–18051 Rostock, Germany*

Received 25 April 1997, accepted 12 May 1997

It is shown that least-squared inversion is a suitable method for reconstructing the density matrix from measurable time-dependent quantities. It enables one to take into account various specific features of experiments, such as limited sets of data and data smearing owing to limited resolution. The method can be used to reconstruct the quantum state of various systems, such as harmonic and anharmonic oscillators including molecular vibrations in vibronic transitions and damped motion. To illustrate it, we consider the reconstruction of the density matrix of a damped harmonic oscillator.

1. Introduction

Tomographic methods have been very fruitful for reconstructing the quantum state of light [1] and matter systems, such as molecular vibrations [2], the transverse motion of an atom beam [3], and the center-of-mass motion of trapped ions [4]. Usually undamped harmonic oscillators are considered and the quantum state is reconstructed from the quadrature-component statistics. However, in many physical systems anharmonic and damped motions are observed. A first attempt has been made to reconstruct the quantum state of anharmonic molecular vibrations using time-resolved fluorescence spectroscopy [5]. It has been shown that the density matrix can be obtained by inversion of high-dimensional systems of linear equations. An approach has been given in [6], extending the pattern-function formalism to more general than harmonic potentials and reconstructing the density matrix from the time-dependent position distribution.

There have been a number of open questions, such as those of the determination of suitable sampling functions mapping the measured data onto the density matrix, the

[1] Presented at the **Fifth Central-European Workshop on Quantum Optics**, Prague, Czech Republic, April 25 - 28, 1997

[2] Permanent address: Palacký University, Faculty of Natural Sciences, Svobody 26, 77146 Olomouc, Czech Republic; e-mail: opatrny@risc.upol.cz

0323-0465/97

choice of optimum observational times, and the inclusion into the scheme of damping effects and data smearing. Some of the questions have been addressed in [7]. For all the systems mentioned the general problem to be solved is the inversion of linear equations that relate the measured quantities to the density-matrix elements of the system under study. This can be done in a very efficient way using the least-squares method, which has already been applied to various quantum-state reconstruction problems, such as the reconstruction of the quantum state of cavity fields [8], vibrations of trapped ions [9], and optical field by balanced [10] and unbalanced [11] homodyning. Here we want to show the applicability of the least squares inversion to the state reconstruction of a damped oscillator.

2. Basic equations

Let us consider a quantum-mechanical system and assume that at some initial time it is prepared in a state with density matrix $\varrho_{n,n'} = \langle n|\hat{\varrho}|n'\rangle$, where $|n\rangle$ are the energy eigenstates of the system Hamiltonian. Further, let us assume that there is a measurable time-dependent (probability) distribution $p(x,t)$ of a quantity x that can be given by a linear combination of all density-matrix elements $\varrho_{n,n'}$ that are initially excited, with linearly independent coefficient functions $S_{n,n'}(x,t)$,

$$p(x,t) = \sum_{n,n'} S_{n,n'}(x,t)\,\varrho_{n,n'}. \tag{1}$$

When we consider, e.g., a particle that moves in a potential well and is initially prepared in a bound state (e.g., a molecular vibration in a vibronic system below the dissociation level), only the discrete part of the energy spectrum is excited ($n = 1, 2, 3, \ldots$). For the sake of transparency, in what follows we restrict attention to discrete spectra. However, replacing in Eq. (1) the sums with integrals (or combinations of sums and integrals), excitations of continuous parts of the spectrum can be treated accordingly. For any physical state the density-matrix elements $\varrho_{n,n'}$ must eventually decrease indefinitely with increasing $n(n')$. Therefore it follows that the expression on the right-hand side of Eq. (1) can always be approximated to any desired degree of accuracy by setting $\varrho_{n,n'} \approx 0$ for $n(n') > n_{\max}$, if $n_{\max}$ is suitably large.

From the assumptions made it is clear, that Eq. (1) can, in principle, be inverted in order to obtain the quantum state (at time $t = 0$) from the measured function $p(x,t)$. Direct application of least-squares inversion to Eq. (1) yields [7]

$$\tilde{\varrho}_{n,n'} = \int_X dx \int_T dt\, K_{n,n'}(x,t)\, p(x,t), \tag{2}$$

where the sampling function is given by

$$K_{n,n'}(x,t) = \sum_{m,m' \le n_{\max}} F_{n,n';m,m'}\, S^*_{m,m'}(x,t) \tag{3}$$

and $\mathbf{F} = \mathbf{G}^{-1}$, with the matrix $\mathbf{G}$ being defined by

$$G_{m,m';n,n'} = \int_X dx \int_T dt\, S^*_{m,m'}(x,t) S_{n,n'}(x,t). \tag{4}$$

Here T and X, respectively, characterize the measurement time and the size of the interval in which x is measured. The symbol $\tilde{\varrho}_{n,n'}$ is introduced to distinguish between the reconstructed density matrix and the exact one ($\tilde{\varrho}_{n,n'} \to \varrho_{n,n'}$ for $n_{\max} \to \infty$).

3. Damped systems

It is worth noting that Eq. (2) can also be applied to damped systems. Let us suppose that the density matrix evolves according to some master equation $\dot{\hat{\varrho}} = \hat{\mathcal{L}}\hat{\varrho}$, where $\hat{\mathcal{L}}$ is a linear superoperator,

$$\hat{\mathcal{L}}\hat{\varrho} = \frac{1}{i\hbar}[\hat{H}, \hat{\varrho}] + \hat{\mathcal{R}}\hat{\varrho}, \tag{5}$$

with $\hat{\mathcal{R}}$ describing the effect of (Markovian) damping. Since the solution of this master equation can always be represented in the form of

$$\varrho_{m,m'}(t) = \sum_{n,n'} U_{m,m';n,n'}(t)\,\varrho_{n,n'}, \tag{6}$$

$\varrho_{n,n'} \equiv \varrho_{n,n'}(0)$, the function $S_{n,n'}(x,t)$ in Eq. (1) can be given by

$$S_{n,n'}(x,t) = \sum_{m,m'} S_{m,m'}(x)\, U_{m,m';n,n'}(t), \tag{7}$$

where $S_{n,n'}(x) \equiv S_{n,n'}(x,0)$ determines, according to Eq. (1), the initial distribution $p(x,0)$. To demonstrate the method, let us consider a one-dimensional harmonic oscillator ($\omega = 1$) undergoing energy relaxation and assume that $p(x,t)$ is the time-dependent position distribution. In this case we have $S_{n,n'}(x) = \psi_n(x)\psi_{n'}(x)$, with $\psi_n(x) = (\sqrt{\pi}2^n n!)^{-1/2}\exp(-x^2/2)\mathrm{H}_n(x)$ being the harmonic-oscillator energy eigenfunctions, and $\hat{\mathcal{R}}\hat{\varrho} = -\beta(a^\dagger a\varrho + \varrho a^\dagger a - 2a\varrho a^\dagger)$ (β, damping constant). Solving the master equation yields $U_{m,m';n,n'}(t) = \sum_{k,l=0}^{\infty} \delta_{m+k,m'}\delta_{m+l,n}\delta_{m'+l,n'}U_{m,k,l}(t)$ for $m' \ge m$, where

$$U_{m,k,l}(t) = \frac{1}{l!}\left[\frac{(m+l)!(m+l+k)!}{m!(m+k)!}\right]^{1/2} e^{ikt}e^{-\beta(2m+k)t}\left(1 - e^{-2\beta t}\right)^l, \tag{8}$$

and $U_{m',m;n',n}(t) = U^*_{m,m';n,n'}(t)$.

(a) (b)

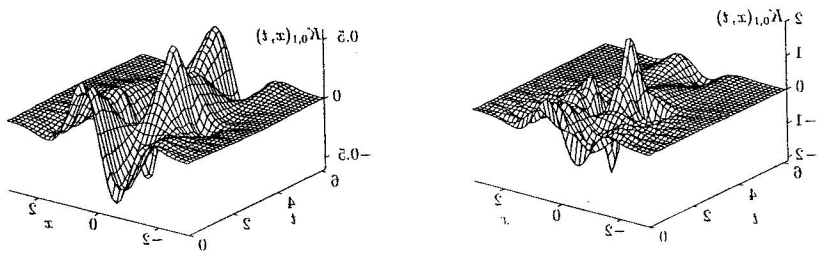

Fig. 1. Sampling function $K_{0,1}(x,t)$ for reconstructing the real part of the density-matrix element $\varrho_{0,1}$ (at time $t = 0$) of a damped harmonic oscillator for $n_{\max} = 12$ and two values of the damping constant [$\beta = 0.03$ (a), $\beta = 0.08$ (b)].

Examples of the sampling function (3) are plotted in Fig. 1, and results of the density-matrix reconstruction are shown in Fig. 2. In our computer experiments we have assumed that 10^6 events are recorded at each of 42 times equidistantly distributed over a 2π interval. For small damping [Fig. 1a] the sampling function is essentially the same as the analytical result for an undamped oscillator (see, e.g., [6]). With increasing

damping it reaches larger absolute values, which (for chosen number of events) implies an increase of the statistical error. Whereas for short times it can be highly structurized, for long times it becomes more and more structureless due to damping [Fig. 1b]. For larger damping a larger number of events must be recorded to keep the statistical error sufficiently small. Using the same amount of data for an undamped oscillator reduces the error such that it cannot be resolved on the scale used in Fig. 2b. In summary we have shown that the density matrix of a damped harmonic oscillator can be reconstructed with a sufficiently good precision.

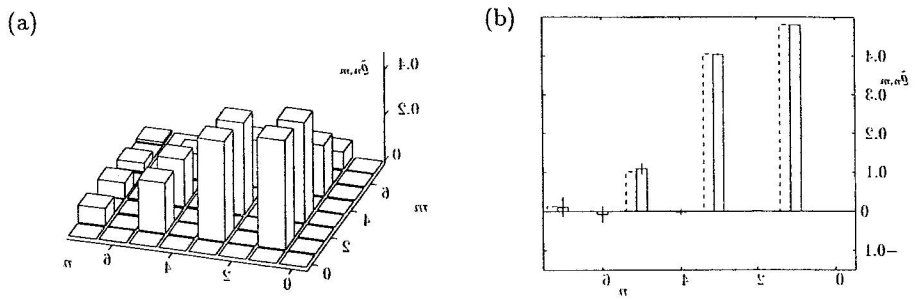

Fig. 2. Reconstruction of the density-matrix elements of a damped harmonic oscillator ($\beta = 0.08$), which is initially prepared in a state $|\psi\rangle = A(|\alpha\rangle - |-\alpha\rangle)$ ($\alpha = 1.5$), from the data in a simulated position measurement for $T = 2\pi$; (a) reconstructed density matrix (real part), (b) comparison of the reconstructed diagonal elements of damped (full lines) and undamped (dashed lines) oscillators. The error bars indicate the statistical error.

Acknowledgement This work was supported by the Deutsche Forschungsgemeinschaft.

References

[1] D.T. Smithey, M. Beck, M.G. Raymer, A. Faridani: *Phys. Rev. Lett.* **70** (1993) 1244; S. Schiller, G. Breitenbach, S.F. Pereira, T. Müller, J. Mlynek: *ibid.* **77** (1996) 2933

[2] T.J. Dunn, I.A. Walmsley, S. Mukamel: *Phys. Rev. Lett.* **74** (1995) 884

[3] C. Kurtsiefer, T. Pfau, J. Mlynek: *Nature* **386** (1997) 150

[4] S. Wallentowitz, W. Vogel: *Phys. Rev. Lett.* **75** (1995) 2932

[5] M. Shapiro: *J. Chem. Phys.* **103** (1995) 1748

[6] U. Leonhardt, M.G. Raymer: *Phys. Rev. Lett.* **76** (1996) 1989; Th. Richter, A. Wünsche: *Phys. Rev. A* **53** (1996) R1974

[7] T. Opatrný, D.-G. Welsch, W. Vogel: *http://xxx.lanl.gov/abs/quant-ph/9703026*

[8] P.J. Bardroff, E. Mayr, W.P. Schleich, P. Domokos, M. Brune, J. M. Raimond, S. Haroche: *Phys. Rev. A* **53** (1996) 2736

[9] P.J. Bardroff, C. Leichtle, G. Schrade, W.P. Schleich: *Phys. Rev. Lett.* **77** (1996) 2198; D. Leibfried, D.M. Meekhof, B.E. King, C. Monroe, W.M. Itano, D.J. Wineland: *Phys. Rev. Lett.* **77** (1996) 4281

[10] S.M. Tan: *J. Mod. Optics*, to be published

[11] T. Opatrný, D.-G. Welsch: *Phys. Rev. A* **55** (1997) 1462; T. Opatrný, D.-G. Welsch, S. Wallentowitz, W. Vogel: *J. Mod. Optics*, to be published