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COMMUTATOR OF THE PHOTON NUMBER AND HERMITIAN PHASE OPERATORS[1]

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The "discrete-time" dynamics is addressed in the framework of the finite photon number and discrete phase formalism. Moreover, the approximations of number-phase commutator are compared.

1. Introduction

The importance of the commutator in the formalism of quantum optics derives from two sources. First, the commutator enters the Heisenberg-Robertson uncertainty relation [1]. Second, it is connected to the dynamics of a system. The role of the commutator in the Heisenberg-Robertson uncertainty relation is almost never doubted. The derivation of this uncertainty relation comprises an introduction of operator variances and covariance which, as a quasiclassical notion, is obtained through symmetrization of the operator product. A necessary antisymmetric term can be expressed by the commutator. One can doubt the introduction of an operator variance (e.g., the quantum phase variance) rather than the commutator in the uncertainty relation.

In the dynamics, the commutator enters an evolution equation, because the time on which the desired solutions of the evolution equation depend, is a continuous parameter. In the harmonic oscillator the time and the phase are linked together. Although we share the opinion that the continuous time and a discrete phase can co-exist in the harmonic oscillator, we are tempted by the idea that the discrete phase could also mean a "discrete" time.

The discrete phase occurs along with a finite range of the photon number. In this situation we can replace the usual commutator by the group-theoretical commutator, which can enter the Weyl commutation relation. The Weyl commutation relation can become a Heisenberg commutation relation between the operators which generate arbitrarily small shifts. Since the exponential phase and angle operators cause the least

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shift of the action and angular momentum by the Planck constant h and $\hbar = h/2\pi$, respectively, the early recommendation follows to restrict oneself to the commutation relation between the action and the operator of the least shift, but not directly between the action and the quantum phase [2]. We can thus avoid an "ugly" commutator. But what to do in case this singular commutator is of appeal [3,4]? Besides, a difference between the action operator and the displaced action operator better corresponds to a hybrid of Lie-algebraic and group-theoretical commutators.

Barnett and Pegg [5] intended to characterize the limit behaviour of the commutator of the number operator and their phase operator in the finite photon number and discrete phase formalism. Were they to modify London's recommendation and to study the commutation relation between the operator of the least shift in the discrete phase (the phase step) and their phase operator? Or, as remarked, they could study the difference between the phase operator and the displaced phase operator. As stated above, the uncertainty relation does not allow us to doubt the commutator and Barnett and Pegg [5] have expressed only the commutator of the number operator and their phase operator indeed.

2. Hermitian optical phase operators

Pegg and Barnett [6,7] introduced the Hermitian phase formalism, which is based on the observation that the states with the better-and-better-defined phase can have a simple number state basis representation [8]. Thus they restrict the state space to a finite $(s+1)$-dimensional Hilbert space $\mathcal{H}_s$ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. The identity operator appropriate to this space and the restricted number operator are

$$\hat{1}_s = \sum_{n=0}^{s} |n\rangle\langle n|, \quad \hat{n}_s = \sum_{n=0}^{s} n|n\rangle\langle n|. \tag{1}$$

In the space $\mathcal{H}_s$ Pegg and Barnett define a complete orthonormal system of phase states

$$|\theta_m; s\rangle = (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \quad m = 0, \ldots, s, \tag{2}$$

where the values of θ_m are given by

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}, \quad m = 0, \ldots, s. \tag{3}$$

The value of θ_0 can be chosen as any real number. An Hermitian optical phase operator is constructed directly from the orthonormal phase states [6]

$$\hat{\phi}_{\theta_0,s} = \sum_{m=0}^{s} \theta_m |\theta_m; s\rangle\langle\theta_m; s|. \tag{4}$$

Further operators are constructed indirectly from the orthonormal phase states, e. g. $\exp(i\hat{\phi}_{\theta_0,s})$, $\cos(\hat{\phi}_{\theta_0,s})$, and $\sin(\hat{\phi}_{\theta_0,s})$. The PB prescription is to evaluate any observable

of interest in the finite basis (2) and only after that take the limit $s \to \infty$. It is useful to introduce the values of θ_m for any real number m, but let us consider first any integer m. The values of θ_m then form a second-kind $(s+1)$-periodic sequence,

$$\theta_{m+s+1} = \theta_m + 2\pi, \quad m \text{ an integer}, \tag{5}$$

but the phase states are still first-kind $(s+1)$-periodic,

$$|\theta_{m+s+1}; s\rangle = |\theta_m; s\rangle, \quad m \text{ an integer}. \tag{6}$$

The direct construction of the Hermitian phase operator (4) can be generalized,

$$\hat{\phi}_{\theta_a,s} = \sum_{m=a}^{a+s} \theta_m |\theta_m; s\rangle\langle\theta_m; s|, \quad a \text{ an integer}, \tag{7}$$

but, in fact, no new operator has been introduced. The generic Hermitian phase operator $\hat{\phi}_{\theta_a,s}$ is second-kind $(s+1)$-periodic in dependence on a,

$$\hat{\phi}_{\theta_{a+s+1},s} = \hat{\phi}_{\theta_a,s} + 2\pi\hat{1}_s. \tag{8}$$

The appropriate matrix elements read

$$\begin{aligned} \langle n'|\hat{\phi}_{\theta_a,s}|n\rangle &= i\frac{\pi}{s+1}\exp\left[-i(n-n')\theta_{a+s+1/2}\right]\csc[(n-n')\pi/(s+1)], \ n' \neq n, \\ \langle n|\hat{\phi}_{\theta_a,s}|n\rangle &= \theta_{a+s/2}. \end{aligned} \tag{9}$$

In [9] a 2π-periodic function $\phi_\alpha(\varphi)$ has been defined, $\phi_\alpha(\varphi) = \varphi$ in the interval $(\alpha, \alpha+2\pi)$ and $\phi_\alpha(\alpha) = \pi + \alpha$. Since $\phi_\alpha(\varphi) = \frac{1}{2}[\phi_\alpha(\varphi - 0) + \phi_\alpha(\varphi + 0)]$ everywhere, it can be expanded in a Fourier series. In [9, 10] the function $\phi_\alpha(\varphi)$ allowed us to define the operators

$$\hat{\phi}_{\alpha,\theta_0,s} = \sum_{m=0}^{s} \phi_\alpha(\theta_m)|\theta_m; s\rangle\langle\theta_m; s|. \tag{10}$$

The generic operator $\hat{\phi}_{\alpha,\theta_0,s}$ is second-kind 2π-periodic in dependence on α,

$$\hat{\phi}_{\alpha+2\pi,\theta_0,s} = \hat{\phi}_{\alpha,\theta_0,s} + 2\pi\hat{1}_s. \tag{11}$$

Now a whole continuum of these operators are just the PB Hermitian phase operators, but we obtain new Hermitian phase operators for discrete values of the variable α. It holds that

$$\hat{\phi}_{\alpha,\theta_0,s} = \hat{\phi}_{\theta_a,s} \text{ for } \alpha \in (\theta_{a-1}, \theta_a), \tag{12}$$

but for $\alpha = \theta_{a-1}$ the appropriate matrix elements read

$$\begin{aligned} \langle n'|\hat{\phi}_{\theta_{a-1},\theta_0,s}|n\rangle &= i\frac{\pi}{s+1}\exp[-i(n-n')\theta_{a+s}]\cot[(n-n')\pi/(s+1)], \ n' \neq n, \\ \langle n|\hat{\phi}_{\theta_{a-1},\theta_0,s}|n\rangle &= \theta_{a+(s-1)/2}. \end{aligned} \tag{13}$$

For example, the introduced operator $\hat{\phi}_{\theta_{-1},\theta_0,s}$ has an eigenvalue $\frac{1}{2}(\theta_{-1} + \theta_s)$ in the phase state $|\theta_{-1}; s\rangle = |\theta_s; s\rangle$.

3. Commutators

In [5] the limit behaviour of the commutator $[\hat{n}_s, \hat{\phi}_{\theta_0,s}]$ as s tends to infinity has been characterized. There does not exist any function $f(\varphi)$ to enable one to express this commutator as $f(\hat{\phi}_{\theta_0,s})$ exactly, although approximations are possible. The requirement that an approximation should comprise the phase state $|\theta_0; s\rangle$, leads to the relation

$$-i[\hat{n}_s, \hat{\phi}_{\theta_0,s}] \approx \hat{1}_s - (s+1)|\theta_0; s\rangle\langle\theta_0; s|$$
$$= \left[\hat{\phi}_{\theta_0,s} - \exp\left(i\frac{2\pi}{s+1}\hat{n}_s\right)\hat{\phi}_{\theta_0,s}\exp\left(-i\frac{2\pi}{s+1}\hat{n}_s\right)\right]\frac{s+1}{2\pi}. \tag{14}$$

There exists a function $f(\varphi)$ to enable us to approximate the commutator,

$$\hat{1}_s - (s+1)|\theta_0; s\rangle\langle\theta_0; s| = f(\hat{\phi}_{\theta_0,s}), \tag{15}$$

where

$$f(\varphi) = \left[\phi_{\theta_{-1/2}}(\varphi) - \phi_{\theta_{-1/2}}\left(\varphi - \frac{2\pi}{s+1}\right)\right]\frac{s+1}{2\pi}. \tag{16}$$

Instead, we require that the approximation should use the phase state $|\theta_{-1/2}; s\rangle$. Thus we obtain the relation

$$-i[\hat{n}_s, \hat{\phi}_{\theta_0,s}] \approx \hat{1}_s - (s+1)|\theta_{-1/2}; s\rangle\langle\theta_{-1/2}; s|$$
$$= \left[\exp\left(-i\frac{\pi}{s+1}\hat{n}_s\right)\hat{\phi}_{\theta_0,s}\exp\left(i\frac{\pi}{s+1}\hat{n}_s\right) - \exp\left(i\frac{\pi}{s+1}\hat{n}_s\right)\hat{\phi}_{\theta_0,s}\exp\left(-i\frac{\pi}{s+1}\hat{n}_s\right)\right]\frac{s+1}{2\pi}. \tag{17}$$

There exists a function $f(\varphi)$ to express the approximate commutator,

$$\hat{1}_s - (s+1)|\theta_{-1/2}; s\rangle\langle\theta_{-1/2}; s| = f(\hat{\phi}_{\theta_{1/2},s}), \tag{18}$$

where

$$f(\varphi) = \left[\phi_{\theta_{-1/2}}\left(\varphi + \frac{\pi}{s+1}\right) - \phi_{\theta_{-1/2}}\left(\varphi - \frac{\pi}{s+1}\right)\right]\frac{s+1}{2\pi}. \tag{19}$$

The commutator $[\hat{n}_s, \hat{\phi}_{\theta_{-1},\theta_0,s}]$ and its approximations can be obtained similarly.

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