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GENERALIZED ONE-DIMENSIONAL REPRESENTATION IN QUANTUM OPTICS[1]

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Continuous one-dimensional superposition of coherent states along certain curves in the alpha plane proved to be a convenient representation of several important states in quantum optics. It is shown that the weight function of the superposition can be found provided the Fock state expansion of the state is given. The weight function associated with the phase optimized state is presented.

1. Introduction

When we perform quantum mechanical calculations, a good choice of the basis vector set in the Hilbert space of the system may simplify the calculations. For example, if we calculate the mean value of an operator, it is advantageous to expand in series the state vector with respect to the eigenstates of the operator, since the operator is diagonal in this basis.

In quantum optics the Hilbert space of the light field is that of the harmonic oscillator. One preferred basis is the Fock basis, the eigenstates of the Hamiltonian of the free electromagnetic field. It was shown in Ref. [1], that coherent states of the harmonic oscillator form an overcomplete basis set in the Hilbert space of the oscillator. Coherent states are eigenstates of the field annihilation operator. An overcomplete basis set is a set of basis vectors in the Hilbert space which is more numerous than the smallest basis which spans the Hilbert space.

Coherent states proved to be a very convenient basis set. Any state $|\Psi\rangle$ in the Hilbert space can be developed into a continuous superposition of coherent states

$$|\Psi\rangle = |f(\alpha)\rangle = \int d^2\alpha f(\alpha)|\alpha\rangle, \tag{1}$$

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where α runs over the complex plane. The definition of $f(\alpha)$ is not unique because of the overcompleteness of the coherent states.

There are some methods for finding a reduced set of coherent states which is still complete and easy to calculate with it [2]. It is pointed out in Ref. [3] that quadrature squeezed vacuum states can be represented by a Gaussian superposition of coherent states along a straight line in the α plane:

$$|F\rangle = \int_{-\infty}^{\infty} d\alpha F(\alpha)|\alpha\rangle$$
$$F(\alpha) = \pi^{-1/2}\gamma^{-1}(1+\gamma^2)^{1/4}\exp(-\alpha^2/\gamma^2). \quad (2)$$

It was shown in Refs. [4,5] that any quadrature squeezed coherent state in one and two dimension can be obtained by superposing coherent states along a straight line in the α plane with an appropriate weight function. In recent paper [6], a complete orthonormal basis set on a straight line has been presented, which make it possible to obtain the coherent-state expansion on a straight line for a given sate. In the following paper [7], a systematic construction of the circle representation of a quantum state has been obtained.

In this paper we generalize the concept of one-dimensional representation. A systematic method is shown for finding the weight function of the continuous superposition provided the Fock state expansion of the state is known. As an example the infinite straight line superposition weight function of the phase optimized state is presented.

2. Generalized one-dimensional representation

In this section we will prove that any state in the Hilbert space of the harmonic oscillator can be approximated with arbitrary precision by continuous one-dimensional superposition of coherent states:

$$|\Psi\rangle = \int_\Gamma d\alpha g(\alpha)|\alpha\rangle. \quad (3)$$

Here the domain Γ is a continuous curve in the α plane. The approximate nature of this relation will be explained later.

Now, a systematic method is derived to find the weight function $g(\alpha)$ in the expansion Eq. (3). In the following, it will be useful to introduce orthonormal polynomials $P_n(\alpha)$

$$\int_\Gamma d\alpha P_n(\alpha)P_m(\alpha)w(\alpha) = \delta_{nm}, \quad (4)$$

where $w(\alpha)$ is a weight function. We also assume that these polynomials form a complete set, moreover for any functions $f(x)$ for which $(f.f)$ is finite, the following polynomial approximation is valid

$$f(\alpha) = \sum_{n=0}^{N} f_n P_n(\alpha), \quad f_n = \int_\Gamma d\alpha' w(\alpha')f(\alpha')P_n(\alpha'). \quad (5)$$

We note that most of the orthogonal polynomials occurring in mathematical physics satisfy these conditions. We redefine the function $g(\alpha)$ in Eq. (3)

$$G(\alpha) = \frac{g(\alpha)e^{-|\alpha|^2/2}}{w(\alpha)}, \quad (6)$$

which yields the following equation for $|\Psi\rangle$:

$$|\Psi\rangle = \int_\Gamma d\alpha\ w(\alpha)G(\alpha)\sum_{n=0}^{N}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad (7)$$

where the summation is truncated at an arbitrarily large but finite value N to prevent problems with the change of the order of integration and summation. The approximate nature of the expansion (3) comes from this truncation. A little rearrangement in Eq. (7) yields a unique connection between the Fock state expansion coefficients of the state $|\Psi\rangle$ and the integrals of the α^m weighted with the weight function $G(\alpha)$. The powers of α are polynomials which form a basis set in the domain Γ. The set of polynomials $\{\alpha^m\}_{m\in[0,N]}$ can be orthogonalized in the domain Γ with respect to a weight function $w(\alpha)$. The resulting polynomials are the previously introduced polynomials $P_m(\alpha)$ in Eq. (4). The powers of α can be expressed with these polynomials.

Inserting the polynomial expansion of α^m into Eq. (7) we obtain

$$\tilde{c}_m = \sum_{n=0}^{m}\{P^{-1}\}_{mn}\int_\Gamma d\alpha\ w(\alpha)G(\alpha)P_n(\alpha), \quad (8)$$

where $\tilde{c}_m = c_m\sqrt{m!}$. The matrix P^{-1} is the inverse of the lower triangle matrix P, which is composed of the coefficients of the polynomials $P_n(\alpha)_{n=0..N}$. The diagonal elements P_{mm} are nonzero, since these are the coefficients of the highest power of α in the polynomials $P_m(\alpha)$. This feature ensures that the matrix P can be inverted, since $\det P = \prod_{m=0}^{N} P_{mm}$ is nonzero. The integral in Eq. (8) is the generalized Fourier expansion of the function $G(\alpha)$. We can easily invert Eq. (8) to obtain the Fourier coefficients. From the Fourier coefficients, recalling the definition of $G(\alpha)$ in Eq. (6), and the completeness of $P_n(\alpha)$ Eq. (5), finally we arrive to the following form of the weight function $g(\alpha)$:

$$g(\alpha) = e^{\alpha^2/2}w(\alpha)\mathbf{P}(\alpha)P\tilde{c}, \quad \alpha\in\Gamma, \quad (9)$$

here vector $\mathbf{P}(\alpha) = \{P_0(\alpha), P_1(\alpha), ..., P_N(\alpha)\}$ and vector $\tilde{c} = \{\tilde{c}_0, \tilde{c}_1, ..., \tilde{c}_N\}$.

3. Discrete representation of the phase optimized state

To demonstrate how our method works, we present the one-dimensional weight function of the phase optimized state defined in Ref. [8]. The Fock state expansion of the state is

$$|\Psi\rangle = \mathcal{N}\sum_{k=0}^{\infty} Ai(a(k+b)+b_0)|k\rangle, \quad (10)$$

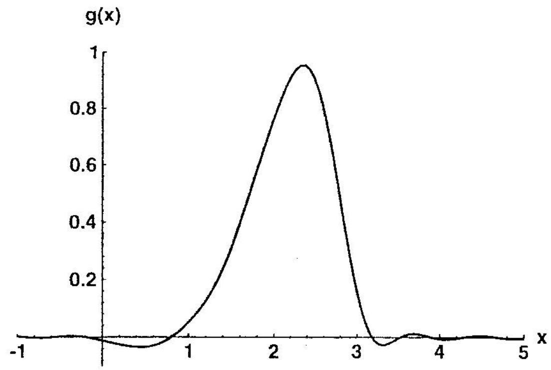

Fig. 1. The plot of the weight function $g(\alpha)$ for the phase optimized state.

where Ai is the Airy function, $b_0 = -2.34$ is its first zero value, $0 < b < 1$ is a shift parameter, parameter a presets the mean photon number, $\mathcal{N}$ is a normalization constant.

To find a one-dimensional coherent state representation of the state Eq. (10), we choose the domain $[-\infty, \infty]$, and the Hermite polynomials to construct the weight function $g(\alpha)$. Inserting the Fock state coefficients of the phase optimized state Eq. (10) to the formula Eq. (9) we obtain an expression which can be summed numerically. The plot of the resulting weight function $g(\alpha)$ is in Fig. 1.

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