NONCLASSICAL STATES WITH MAXIMUM ENTROPY[1]

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We investigate nonclassical single-mode fields with the property that only number-states (Fock-states) differing by a multiple of a certain integer k ($k \geq 2$) are allowed to be occupied and that the entropy of the fields takes its maximum value. For special initial conditions, such maximum-entropy states are the stationary solutions of a master equation which takes into account k-quantum absorption as well as k-quantum emission processes only. The number-probability distribution of the even-number maximum-entropy state arising for $k = 2$ has a close resemblance to that of a squeezed-vacuum state.

1. Introduction

It is well known that the density operator of a radiation field being in thermodynamical equilibrium with its surrounding can be determined by maximizing its entropy under the constraint that the mean energy (or the mean photon number, respectively) is kept constant. The expression which is valid for the thermal density operator also holds for the density operator of a chaotic radiation field that results from the contributions of many statistically independent classical light sources [1]. In this contribution we generalize the concept of the maximum-entropy state of a single quantized harmonic oscillator of frequency ω to nonclassical k-quantum maximum-entropy states ($k = 2, 3, \ldots$) that arise when only energy eigenstates are allowed to be occupied the energy of which differs by a multiple of $k\hbar\omega$. These states do not posses a positive-definite P-representation, i.e. they are intrinsically nonclassical.

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2. Properties of the states

We can discriminate k different kinds of k-quantum maximum-entropy states which we label by the integer parameter q where $0 \leq q \leq k-1$. The diagonal density-matrix elements obey the equation

$$\rho_{nn}^{(k,q)} = \begin{cases} 0 & \text{for} \quad n \neq mk+q \\ p_{mk+q} & \text{for} \quad n = mk+q \end{cases} \tag{1}$$

where $m = [n/k]$ is the largest integer that does not exceed n/k. The von-Neumann-entropy, which takes its maximum value in the equilibrium state, can be written as

$$S^{(k,q)} = -\sum_{n=0}^{\infty} \rho_{nn}^{(k,q)} \ln(\rho_{nn}^{(k,q)}) = -\sum_{m=0}^{\infty} p_{mk+q} \ln(p_{mk+q}). \tag{2}$$

In order to obtain the number-probability distribution of the maximum-entropy states, we have to maximize the expression (2) under the constraints $\sum_m p_{mk+q} = 1$ and $\sum_m (mk+q) p_{mk+q} = \overline{n}$ with $\overline{n}$ being the mean number of quanta. Using the method of Lagrange multipliers we find that the entropy $S^{(k,q)}$ takes its maximum value when

$$p_{mk+q} = \frac{k}{\overline{n}-q+k} \left(\frac{\overline{n}-q}{\overline{n}-q+k} \right)^m . \tag{3}$$

The structure of Eq. (3) is analogous to that of a thermal or chaotic photon-number distribution [1] with mean photon number $(\overline{n}-q)/k$. In the k-quantum maximum-entropy states all higher order photon-number moments can be expressed by the mean photon number [2]. Whether the photon statistics is super-Poissonian or sub-Poissonian, depends on the values of the parameters $\overline{n}$, k and q.

In the special case $k=2$, $q=0$ only Fock-states belonging to even photon numbers are occupied. As can be seen from Fig. 1, the photon-number distribution of the even-number two-quantum maximum-entropy state has a close resemblance to the recently measured [3] oscillating photon-number distribution of a squeezed-vacuum state arising from spontaneous parametric down conversion. The Q-function $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle/\pi$ of this special maximum-entropy state takes the form

$$Q^{(2,0)}(\alpha) = \frac{1}{\pi} \frac{2e^{-|\alpha|^2}}{\overline{n}+2} \cosh\left(|\alpha|^2 \sqrt{\frac{\overline{n}}{\overline{n}+2}} \right), \tag{4}$$

from which we can derive the Wigner-function by Fourier transformation. In the same way, it is possible to obtain simple analytic expressions for the Q-function and the Wigner-function of the odd-number two-photon maximum-entropy state. It turns out that in the latter case the Wigner-function is strongly negative for $\alpha = 0$ [2].

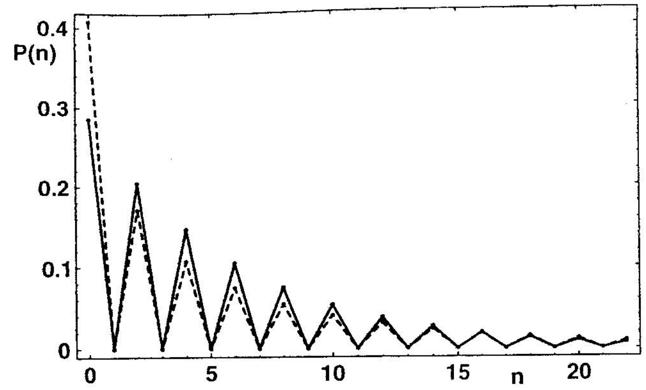

Fig. 1. Number-probability distribution $P(n) = \rho_{nn}$ of an even-number two-quantum maximum-entropy state with mean number of quanta $\overline{n} = 5$ and of a squeezed-vacuum state with the same value of $\overline{n}$. For clearness, the separate dots are connected by a full line (two-quantum maximum-entropy state) and by a dashed line (squeezed vacuum).

3. Generation of the states

We assume that the considered harmonic-oscillator mode interacts with surrounding reservoirs only by k-quantum emission processes (with emission constant β_k) and k-quantum absorption processes (with absorption constant γ_k). This can be described by the master equation

$$\begin{aligned} \dot{\rho} = & -\frac{\beta_k}{2}\left(a^k a^{\dagger k} \rho - 2a^{\dagger k} \rho a^k + \rho a^k a^{\dagger k}\right) \\ & -\frac{\gamma_k}{2}\left(a^{\dagger k} a^k \rho - 2a^k \rho a^{\dagger k} + \rho a^{\dagger k} a^k\right), \end{aligned} \tag{5}$$

In the Fock-representation Eq. (5) yields

$$\dot{p}_n = \frac{(n+k)!}{n!}(\gamma_k p_{n+k} - \beta_k p_n) - \frac{n!}{(n-k)!}(\gamma_k p_n - \beta_k p_{n-k}) \tag{6}$$

for the diagonal elements $\rho_{nn} \equiv p_n$ of the density matrix. The stationary solution of Eq. (6) depends on the initial conditions. If for $k \geq 2$ the initial state fulfills the condition $p^{(i)}_{k[n/k]+q} = 0$ for all $q \neq q_0$, the system evolves towards a steady state which is a k-quantum maximum-entropy state with $q = q_0$. The steady-state mean quantum number is then given by [2]

$$\overline{n} = q_0 + \frac{k}{\gamma_k/\beta_k - 1}. \tag{7}$$

In particular, the required initial condition is fulfilled when the system is prepared in a q_0-quantum Fock-state. According to Eq. (7), a steady state can be only reached when the absorption constant γ_k exceeds the emission constant β_k. This is obvious from physical reasons since already in the case $\beta_k = \gamma_k$ the mean number of quanta would grow to infinity due to the excess caused by spontaneous k-quantum emission.

Finally we address the question as to how the k-quantum maximum- entropy states could be produced in a real experiment. With respect to the even-number two-photon maximum-entropy states one could think of radiation fields resulting from the contributions emitted spontaneously by many statistically independent two-photon emitters, in analogy to the classical chaotic fields [1] resulting from spontaneous single-photon emission. However, even if 2-quantum maximum-entropy fields could be produced this way, their detection seems to be extremely difficult because of the different emission angles involved in spontaneous emission.

On the other hand, an interesting possibility for the generation of nonclassical k-quantum maximum-entropy states becomes obvious when the concepts of quantum optics are adopted to the description of the motional quantum state of a trapped ion in a harmonic-oscillator potential. When the ion is irradiated by laser light of frequency ω_L which is tuned to a definite vibrational sideband of the internal electronic transition with resonance frequency ω_0, the center-of-mass motion of the ion is influenced in a specific way due to the recoil experienced in each cycle of absorption and subsequent spontaneous emission. By changing the laser frequencies and intensities it is possible to design different kinds of coupling between the quantized vibrational harmonic-oscillator mode with trap frequency ω and the surrounding reservoir of the vacuum modes of the electromagnetic field [4]. Provided that the ion is excited at the k-th lower motional sideband ($\omega_L = \omega_0 - k\omega$), k phonons are annihilated during each absorption-spontaneous emission cycle of the ion, and for excitation at the k-th upper sideband ($\omega_L = \omega_0 + k\omega$) we have the case of k-phonon emission. When the ion is irradiated by two appropriately detuned incoherent lasers, a combination of k-phonon absorption and k-phonon emission resulting in the master-equation (5) could be experimentally realized. The ratio γ_k/β_k would then be determined by the ratio of the squared Rabi-frequencies or of the laser intensities respectively. In particular, in order to prepare the even-number two-quantum maximum-entropy state the properties of which are shown in Fig. 1, one could start from the motional ground state of the ion.

In summary, we have introduced specific k-quantum maximum-entropy states which posses nonclassical properties and shown that these states could be produced in a real experiment.

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