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FULLY PHASE SPACE APPROACH TO ATOM–FIELD INTERACTIONS[1]

A. Czirják[2], M. G. Benedict[3]
Department of Theoretical Physics, Attila József University,
H-6720 Szeged, Tisza Lajos körút 84-86, Hungary

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We build up in a systematic way the joint Wigner representation for the Dicke-type atom–field interactions. We add certain new results for the systems described by the Jaynes–Cummings model.

1. Introduction

Wigner functions, or more generally quasiprobability density functions and related phase space methods have been proven to be extremely useful in quantum optics [1, 2], regarding the description of the states of an electromagnetic field mode or the vibrational states of molecules. This has been made possible by the smart construction of these functions [3], which allows one to place back the description of a quantum system onto the corresponding classical phase space. However, quantum optics deals with the interaction of atoms with the electromagnetic field. Thus, it is rather natural to ask whether it is possible to incorporate also the atomic part of an interacting system into a phase space representation, that is, to give a fully phase space description of the problem.

2. Fully phase space description

It is a very frequent and common situation in quantum optics, that the interaction between the collection of identical atoms and the field mode is nearly resonant, and the atoms interact with each other only through the field mode. In this case the atoms can be treated as two-level systems, while their collection as an angular momentum. The interacting system can be described by a Dicke-type Hamilton operator [4, 5].

For such systems the adequate tool for the fully phase space representation is the joint Wigner function [6, 7]. We briefly summarize here its basic elements, which are

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[2]E-mail address: czirjak@sol.cc.u-szeged.hu
[3]E-mail address: benedict@physx.u-szeged.hu

based on the definitions of the usual Wigner function [3, 8, 9, 10] and of the Wigner function for angular momentum [11, 12, 13].

The underlying classical phase space is the direct product of a spherical surface with a plane. (These latters are the classical phase spaces for an angular momentum and a field mode, respectively.) If we want to represent an operator A, acting in the state space of the interacting system, by a function over the corresponding phase space, first we expand it in a suitable operator basis. For our systems this basis is constituted by the tensor products of the displacement operators [9] with the multipole operators [12]:

$$\left\{D^{-1}(\beta,\beta^*)\,T_{KQ}\right\}_{\beta\in\mathrm{C},\ \ K=0,1,\ldots,2j,\ \ Q=-K,-K+1,\ldots,K,} \tag{1}$$

where j is the angular momentum quantum number characterizing the atomic system. The expansion of A is the following [9, 12]:

$$A=\sum_{K=0}^{2j}\sum_{Q=-K}^{K}\frac{1}{\pi}\int \mathrm{d}^2\beta\; C_{KQ}^{(A)}(\beta,\beta^*)D^{-1}(\beta,\beta^*)T_{KQ} \tag{2}$$

where the characteristic functions are to be calculated as

$$C_{KQ}^{(A)}(\beta,\beta^*)=\mathrm{Tr}\left(A\,D(\beta,\beta^*)\,T_{KQ}^{\dagger}\right), \tag{3}$$

Now we have to replace the operator basis (1) by a suitable function basis, which, if we want to obtain a Wigner representation, is the set of functions

$$\left\{\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}\,Y_{KQ}(\theta,\phi)\right\}_{\alpha,\beta\in\mathrm{C},\ \ K=0,1,\ldots,2j,\ \ Q=-K,-K+1,\ldots,K,} \tag{4}$$

where $Y_{KQ}(\theta,\phi)$ denote the spherical harmonics. This mapping transforms (2) into the following equation, which is the definition of the joint Wigner function for A:

$$W^{(A)}(\alpha,\alpha^*,\theta,\phi)=\frac{1}{\pi}\sum_{K=0}^{2j}\sum_{Q=-K}^{K}Y_{KQ}(\theta,\phi)\int \mathrm{d}^2\beta\; C_{KQ}^{(A)}(\beta,\beta^*)\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}. \tag{5}$$

In the case of the joint Wigner function for the density operator, $W^{(\rho)}$, it is customary to normalize it in such a way, that its integral over the phase space is unity. This can be done with the definition

$$W^{(\rho)}(\alpha,\alpha^*,\theta,\phi,t)=\sqrt{\frac{2j+1}{4\pi^5}}\sum_{K=0}^{2j}\sum_{Q=-K}^{K}Y_{KQ}(\theta,\phi)\int \mathrm{d}^2\beta\; C_{KQ}^{(\rho)}(\beta,\beta^*,t)\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}. \tag{6}$$

With the help of the joint Wigner functions one can calculate the quantum expectation values by the following integral over the phase space:

$$\mathrm{Tr}(\rho(t)\,A)=\sqrt{\frac{4\pi}{2j+1}}\int \mathrm{d}^2\alpha\int \mathrm{d}\theta\,\sin\theta\int \mathrm{d}\phi\; W^{(\rho)}(\alpha,\alpha^*,\theta,\phi,t)\,W^{(A)}(\alpha,\alpha^*,\theta,\phi). \tag{7}$$

In order to be able to investigate time evolution, we need an equation of motion. For the joint Wigner function of the density operator it was given in [6], treating the interaction with the Jaynes–Cummings model. In [7] a solution was presented assuming the initial state to be a number state for the field and the excited state for the atom.

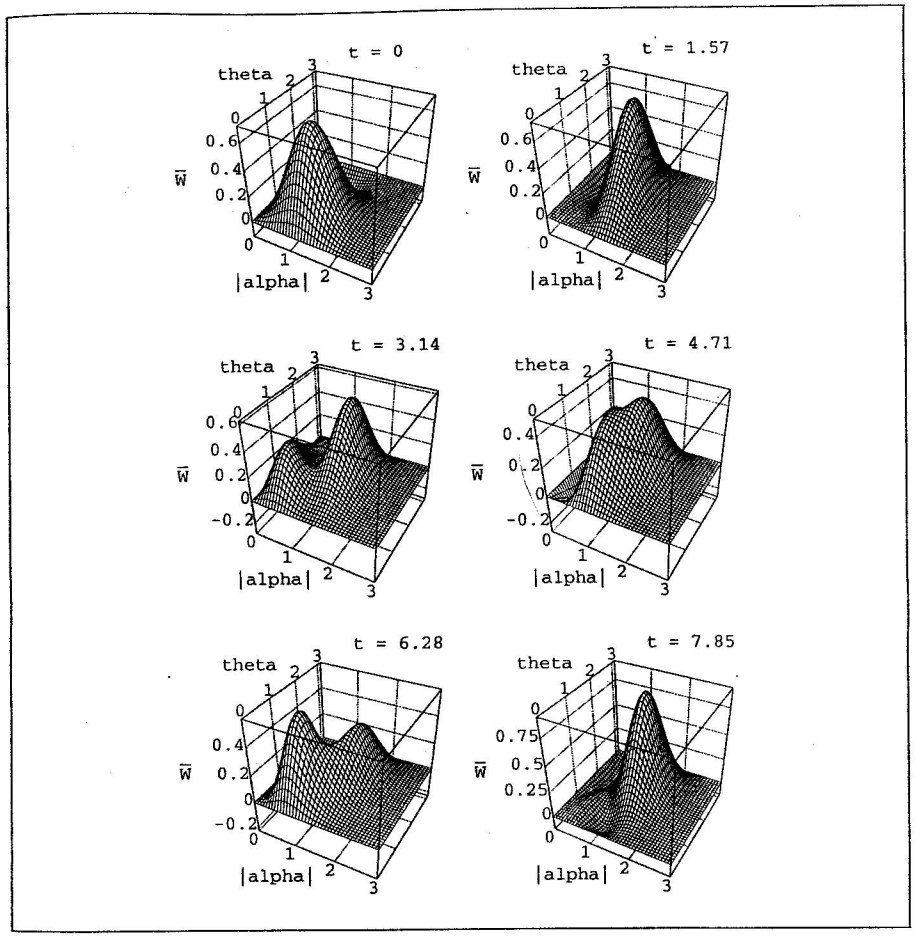

Fig. 1. We plot the reduced joint Wigner function $\overline{W}^{(\rho)}(|\alpha|,\theta,t)$ (see the text for explanation) at six moments of time (in units of g^{-1}), assuming resonance, and the initial state to be a coherent state with amplitude 1 for the field and the excited state for the atom.

3. Application to the Jaynes–Cummings model

Now we give certain new results for the joint Wigner representation applied to the simplest case, when there is only one two-level atom ($j=\frac{1}{2}$) interacting with a nearly resonant field mode. We describe the interaction by the well known Jaynes–Cummings

Hamilton operator [14]:

$$H_{JC} = \hbar\omega_f(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\hbar\omega_a\sigma_z + \hbar g(a\sigma_+ + a^\dagger\sigma_-). \tag{8}$$

According to definition (5), and setting $\alpha = |\alpha|e^{-i\psi}$, the joint Wigner function corresponding to the Hamilton operator (8) is

$$\mathcal{W}^{(H_{JC})}(|\alpha|,\psi,\theta,\phi) = \frac{1}{\sqrt{2\pi}}\left(\omega_f|\alpha|^2 + \frac{\sqrt{3}}{4}\omega_a\cos\theta + \sqrt{3}\,g\,|\alpha|\,\sin\theta\,\cos(\phi-\psi)\right). \tag{9}$$

If we want to visualize the joint Wigner function, we face the following problem: it has four real phase space coordinates, and generally also time, as its arguments. However, we can derive six kinds of reduced joint Wigner function by integrating the joint Wigner function with respect to two coordinates not of interest. It is interesting to keep one field and one atomic variable, since the resulting reduced joint Wigner function still contains information about the corresponding correlations. In Fig. 1 we plot the reduced joint Wigner function

$$\overline{\mathcal{W}}^{(\rho)}(|\alpha|,\theta,t) = \int \mathrm{d}\psi\,|\alpha| \int \mathrm{d}\phi\,\sin\theta\,\mathcal{W}^{(\rho)}(|\alpha|,\psi,\theta,\phi,t) \tag{10}$$

at six moments of time (measured in units of g^{-1}), assuming resonance, and the initial state to be a coherent state with amplitude 1 for the field and the excited state for the atom.

Finally we note, that it is possible to treat these systems with the help of the joint Wigner representation also under more realistic conditions, such as a damping cavity or a position dependent Rabi frequency.

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