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PHASE-INTENSITY UNCERTAINTY RELATION FROM QUASIPROBABILITY DISTRIBUTIONS[1]

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It is well known that measured quasiprobability distributions such as the Q function or the so-called s-parametrized quasiprobability distributions for $s < -1$ that are obtained by using inefficient detectors, allow us to determine a phase distribution as a marginal distribution, in the sense of an operational approach to the quantum phase problem. Starting from the Klein inequality, we first derived a rigorous inequality for the marginals of any positive phase-space distribution. Specifying the latter to the above-mentioned quasiprobability distributions (including the Q- function) and choosing the marginals with respect to phase and amplitude, we reformulated the general entropic uncertainty relation as a phase-intensity uncertainty relation of familiar form. The latter is distinguished by the fact that it holds rigorously, however for unconventionally defined uncertainties. We could show that these new measures of phase and intensity uncertainties actually are in good agreement with the conventional ones. The dependence of the right-hand side of the new uncertainty relation on the parameter s, and hence on the detection efficiency, proves to be extremely simple.

1. Introduction

Certainly, one of the most famous statements in quantum theory is Heisenberg's uncertainty relation for position x and momentum p of a particle which is, in fact, a direct consequence of the commutation relation for the corresponding operators x and p

$$[\hat{p}, \hat{x}] = -i\hat{1} \quad , \quad \Delta x \Delta p \geq \frac{1}{2} \tag{1}$$

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In quantum optics, one identifies position and momentum with the two field quadratures of a single-mode field. Already in the early days of quantum mechanics, a similar reasoning was applied to phase Φ and photon number N

$$[\hat{\Phi}, \hat{N}] = -i\hat{1} \quad , \quad \Delta\Phi\Delta N \geq \frac{1}{2} \tag{2}$$

However, this kind of arguing is dubious, since the commutation relation (2) cannot be fulfilled by a phase operator that is well behaved, the reason being that $\hat{\Phi}$ and $\hat{N}$ are, in a sense, incompatible: The phase operator must have a continuous spectrum from physical reasons, whereas the spectrum of N is discrete and, moreover, has a lower bound. This was actually pointed out by W. Pauli in his famous handbook article on quantum mechanics in the wider context of time operators (of which the phase operator is a special case).

On the other hand, Holevo [1], defining phase uncertainty in an unfamiliar way

$$\Delta\Phi_H^2 = |\langle \exp(i\hat{\Phi})\rangle|^{-2} - 1 \tag{3}$$

succeeded in rigorously deriving the uncertainty relation (2) for phase and photon number. The definition (3) has the advantage that it takes proper account of the periodicity of the phase and, in addition, varies between 0 and ∞. What has been said until now, refers to ideal measurements. However, it is well known that no measuring scheme has been devised for the (ideal) phase, even in the form of a Gedanken experiment. Instead, realistic measurement schemes were specified, in the sense of an operational approach to the quantum phase problem. The first proposal [2] was to amplify the microscopic field under investigation with the help of a (linear) quantum amplifier to a macroscopic level. Then one can measure the phase properties of the amplified field with classical techniques. Later on, L. Mandel et al. [3] proposed and investigated experimentally an eight-port homodyne detection scheme for phase measurement. Theoretically, it was shown (see e.g. [4]) that both schemes yield the same results, provided the local oscillators used in Mandel's setup are strong (laser) fields, the primary result of measurement being the Q function from which the phase distribution is obtained as a marginal distribution.

A further step towards a realistic description of the measuring device is to take into account non-unit detection efficiency which amounts to replace the Q function by the so-called s-parametrized quasiprobability distributions [5]. Here, the parameter s is connected with the detection efficiency η in the simple form $s = -(2-\eta)/\eta$.

Actually, it is not difficult to generalize Heisenberg's uncertainty relation to the case that the uncertainties are calculated with the help of an s-parametrized quasiprobability distribution, the result being [6]

$$\Delta x \Delta p \geq \frac{1-s}{2} \tag{4}$$

In the present paper it is our goal to derive a similar uncertainty relation for realistically measured phase and intensity.

2. Entropic inequalities

We start from the Q function of a given field state. The corresponding distributions for phase and amplitude are given by the marginals of the Q function written in polar coordinates r,φ

$$w(\varphi) = \int_0^\infty r\,dr\,Q(r,\varphi) \tag{5}$$

and

$$w(r) = \int_0^{2\pi} d\varphi\,Q(r,\varphi) \tag{6}$$

We have put $\alpha = r\exp(i\varphi)$ (α complex field amplitude) in order to ensure the correspondence between r^2 and the photon number. Let us now introduce the concept of Wehrl's entropy [7,8]

$$S = -\int_0^{2\pi} d\varphi \int_0^\infty r\,dr\,Q(r,\varphi)\ln Q(r,\varphi) \tag{7}$$

and define, in addition, the following marginal entropies

$$S_\phi = -\int_0^{2\pi} d\varphi w(\phi)\ln w(\varphi) \tag{8}$$

$$S_r = -\int_0^\infty r d r w(r)\ln w(r) \tag{9}$$

We now use Klein's inequality

$$\ln t \leq t - 1 \quad (t > 0) \tag{10}$$

rewritten in the form

$$t = \frac{y}{x} \quad , \quad x(\ln x - \ln y) \geq x - y \tag{11}$$

to derive an inequality between the entropies (7) – (9). On identifying $x = Q(r,\varphi), y = w(\varphi)w(r)$ and integrating over the whole phase space, we readily find the desired relation

$$S_\varphi + S_r \geq S \tag{12}$$

It has been shown that Wehrl's entropy becomes minimum for coherent states in which case it takes the value $1 + \ln\pi$ [8]. We thus end up with the basic inequality

$$S_\varphi + S_r \geq S \geq 1 + \ln\pi \tag{13}$$

This result is readily extended to the more general case of s-parametrized quasiprobability distributions with the result

$$S_\varphi^{(s)} + S_r^{(s)} \geq S^{(s)} \geq 1 + \ln\pi + \ln\frac{1-s}{2} \tag{14}$$

3. Phase-intensity relations

The inequality (14) is equivalent to the relation

$$e^{S_{\varphi}^{(s)}} e^{S_r^{(s)}} \geq e^{S^{(s)}} \geq e\pi \frac{1-s}{2} \tag{15}$$

which, in fact, has the form of an uncertainty relation we are looking for. What still has to be clarified, however, is the connection between the exponentials on the left-hand side and uncertainties. To this end, we study first the special case that the phase distribution is a Gaussian of width $\Delta\varphi \ll 2\pi$. Then a simple calculation gives us the result

$$e^{S_{\varphi}} = (2\pi e)^{\frac{1}{2}} \Delta\varphi \tag{16}$$

As the second observable we consider the intensity (in units of $h\nu$ like the photon number) which is given by $I = r^2$. Noticing the relation $dI = 2rdr$, we see that the intensity distribution $W(I)$ has to be identified with $w(r)/2$. Specializing here also to a narrow Gaussian of width $\Delta I \ll I_0$ (mean intensity), we readily find

$$e^{S_r} = \frac{1}{2}(2\pi e)^{\frac{1}{2}} \Delta I \tag{17}$$

The results (16) and (17) lead us to consider those relations as definitions of new uncertainty measures for phase and intensity. Then it follows from the inequality (15) that the following uncertainty relation holds rigorously

$$\Delta\varphi \Delta I \geq (\pi e)^{-1} e^{S^{(s)}} \geq \frac{(1-s)}{2} \tag{18}$$

This is our main result. It should be noted that in addition to an absolute lower bound there exists also a lower bound $(\pi e)^{-1}\exp(S^{(s)})$ that is specific of the state under consideration. Moreover, it is interesting, however not unexpected, to see that the right-hand sides in Eqs. (4) and (18) are the same, just as in the case of ideal measurements.

We have illustrated our results by some numerical examples. Fig. 1 indicates that Holevo's measure of phase uncertainty adapted to the present situation in the form

$$\Delta\Phi_H^2 = \left| \int_0^{2\pi} w^{(s)}(\varphi) \exp(i\varphi) d\varphi \right|^{-2} - 1 \tag{19}$$

is higher than the variance, whereas our entropic measure is lower. We found that this is, in fact, a general feature exhibited, in particular, by Fig. 2. Further, our numerical analysis revealed that the entropic measure of intensity uncertainty is always smaller than the variance (see Fig. 3). Fig. 4 shows that the entropic phase-intensity uncertainty product for squeezed states comes very close to the absolute lower bound, irrespective of the intensity. Fig. 5 indicates that for a displaced Fock state this product differs noticeably from that based on familiar uncertainty measures. Its value is much greater than the absolute minimum, it comes close, however to the specific lower bound. Finally

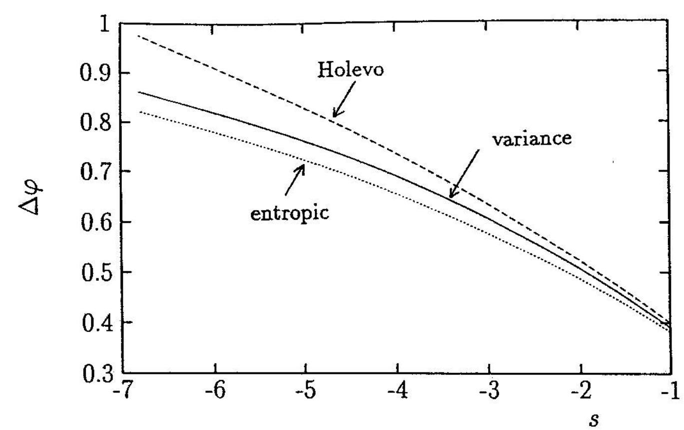

Fig. 1. Different measures of phase uncertainty versus parameter s for a Glauber state with $\alpha = 2$.

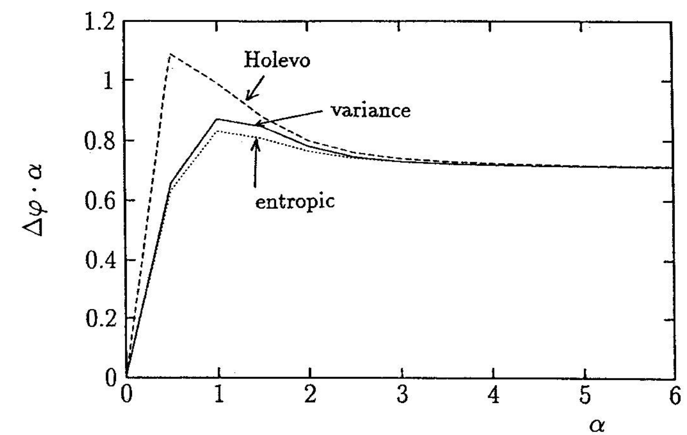

Fig. 2. Different measures of phase uncertainty for a Glauber state and $s = -1$ in dependence on α.

one observes from Fig. 6 that the entropic uncertainty product approaches the other ones for decreasing values of s.

After preparation of this paper, the authors became aware of a theoretical study [9] (cf. also [10]) in which the following entropic uncertainty relation was derived for the

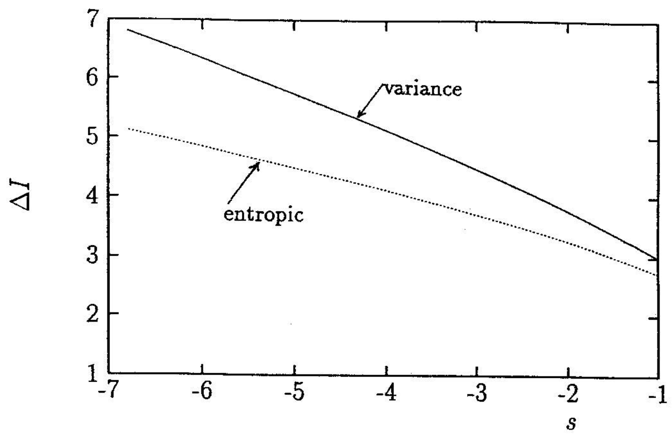

Fig. 3. Different measures of intensity versus s for a Glauber state with $\alpha = 2$.

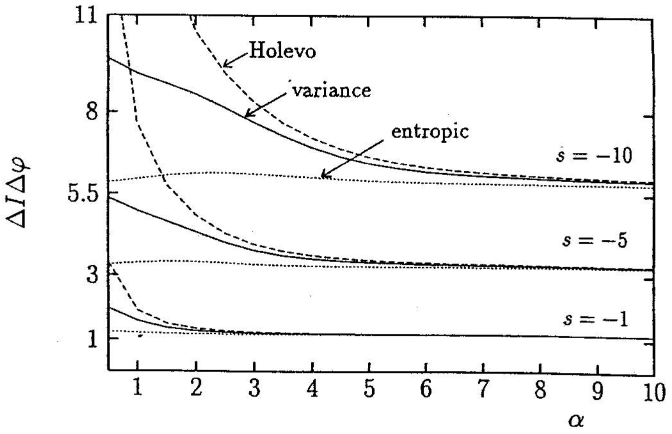

Fig. 4. Intensity-phase uncertainty product for squeezed states with squeezing parameter 3 versus α (displacement). The marks on the right ordinate indicate the respective absolute lower bounds $(1-s)/2$.

case of ideal measurements

$$-\int_0^{2\pi} d\,\Phi W(\Phi)\ln W(\Phi) - \sum_{n=0}^{\infty} W_n \ln W_n \geq \ln(2\pi) \qquad (20)$$

Here, the phase distribution is evaluated as a positive operator-valued measure (POM)

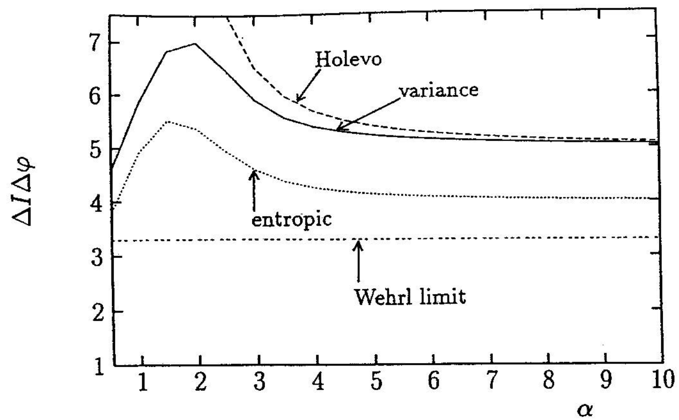

Fig. 5. Same as Fig. 4 for a displaced Fock state with $n = 4$. The Wehrl limit is the specific lower bound $\exp(S/e\pi)$.

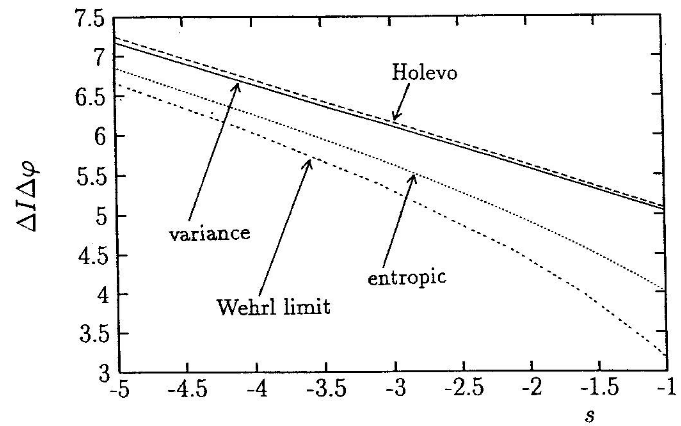

Fig. 6. Intensity-phase uncertainty product for a displaced Fock state with $n = 4$, $\alpha = 10$, versus s.

based on London's phase states, and W denotes the probability to detect n photons. In the light of this paper, our investigation appears to be the natural extension of Eq. (20) to the case of realistic measurements. Our derivation has, however, the advantage that it is rather simple, from the mathematical point of view.

In summary, utilizing the concept of Wehrl's entropy we succeeded in proving rigorously an uncertainty relation for realistically measured phase and intensity. The price we had to pay for this is, similar to Holevo's approach [1], to adopt unfamiliar measures for the uncertainties.

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