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KILLING METRICS IN TWO DIMENSIONS

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In two examples of the 10 orthogonal coordinate systems in 1+1 - dimensional flat spacetime allowing for a separation of the Klein-Gordon equation the associated symmetric Killing tensor fields of order 2 are considered as new metrics replacing the original flat ones. In the arising curved spaces the horizons of the separable coordinate domains become horizons of true curvature singularities; in simple cases a physical interpretation in the framework of 2-dimensional gravity is possible.

1. Introduction

Separable orthogonal coordinate systems on n-dimensional manifolds are characterized by Stäckel systems, that is by linear spaces spanned by n independent symmetric Killing tensors of order 2, including the metric tensor [1]. Killing tensors are characterized by the vanishing of their symmetrized first covariant derivatives,

$$k_{(ij;l)} = 0; \tag{1}$$

the contravariant Killing tensors of a Stäckel system (in two dimensions: g^{ik} and some other tensor k^{ik}) commute with each other in the sense of the Nijenhuis-Schouten bracket [1]

$$[k, g]^{ijk} := k^{ij}_{\ ,l}\, g^{lk} - g^{ij}_{\ ,l}\, k^{lk} = 0 \tag{2}$$

and they are closely related with the horizons of separating coordinate systems in flat space [2].

In flat 1+1 - dimensional space-time, though every Killing tensor may be constructed as a tensor product of Killing vectors, only 2 of 10 separable coordinate systems are associated directly with Killing vectors – the Cartesian and the Rindler system. An important physical significance of Killing vectors is the fact that according to these two

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ordinate systems quantum field theory (QFT) may be established by a positive- and egative-frequency mode decomposition with respect to the one or the other Killing me coordinate [3].

For these reasons it is worthwhile to perform an investigation whether field quanzations based on mode decompositions may be generalized to arbitrary separable co-dinates and whether Killing tensors may be an aid for the construction of quantum ates. Although this work was initially motivated by QFT, geometrical issues of Killing nsors deserve enough attention to do first some purely classical calculations, which ere inspired to a large extent by the Horský-Mitskievich generating conjecture [4] conrning Killing vector fields. Considering the symmetry between g and k in (2) it is mpting to interchange their rôles so that the latter one provides a contravariant metric the curved manifold $\mathcal{M}_2$ (and g^{ik} is an ordinary Killing tensor). The immediately ising question for the geometrical properties and a possible physical interpretation of e objects $(\mathcal{M}_2, k)$ is the contents of this paper.

Separable coordinate vectors being the common eigenvectors of the metric and anher Killing tensor, it is clear that both are diagonal in such coordinate bases. If e contravariant metric components are denoted by g^{ii}, other Killing tensors may be ritten as

$$k^{ii} = \rho_i \, g^{ii}, \qquad i = 0, 1 \tag{3}$$

vithout summation). The ρ_i are functions of the coordinates, which may be obtained solutions of the following set of linear partial differential equations [5]:

$$\partial_i \, \rho_j = (\rho_i - \rho_j) \, \partial_i \ln |g^{jj}|. \tag{4}$$

the following section the Killing tensors associated with two curvilinear orthogonal parating coordinate systems in 2-dimensional Minkowski space are calculated, in secon 3 they are considered as metric tensors on the domains of separable coordinates, hich are bounded by null horizons in $\mathbf{R}_2$, and the resulting curvature scalars are ven. In section 4 $\mathcal{M}_2$ is extended by passing to the cartesian coordinates t and x of e original flat space and the null vector fields of the Killing metrics are determined. pace-time interpretations are given.

2. Separating coordinate systems and Killing tensors

In 1+1 - dimensional Minkowski space there are one elliptic, six hyperbolic, and two rabolic systems [6]. With the exception of one hyperbolic system their domains are en subsets of $\mathbf{R}_2$ bounded by horizons.

1. The elliptic and one of the hyperbolic systems may be treated together, they are alytic continuations of each other. Curvilinear coordinates μ and ν are defined by

$$t^2 = \mu\nu, \qquad x^2 = (\mu - 1)(\nu - 1) \tag{5}$$

th $0 < \nu < \mu < 1$ in the elliptic, and $-\infty < \nu < \mu < 0$ or $1 < \nu < \mu < \infty$ in the perbolic system. Coordinate lines, which are ellipses or hyperbolas, according to the nge of μ and ν, are given by

$$\frac{t^2}{\mu} + \frac{x^2}{1-\mu} = 1 \tag{6}$$

and the same equation in ν. The elliptic system is defined in a square, the hyperbolic one in four space-time wedges attached to the corners of the elliptic coordinate domain (see fig. 1). The two systems have the common horizons $t \pm x = 1$ and $t \pm x = -1$.

Now Killing tensors associated with this system are constructed according to (4) from the flat metric in elliptic (hyperbolic) coordinates:

$$\mathrm{d}s^2 = \frac{\mu - \nu}{4} \left(\frac{\mathrm{d}\nu^2}{\nu(1-\nu)} - \frac{\mathrm{d}\mu^2}{\mu(1-\mu)} \right). \tag{7}$$

The general solution of (4) is given by ($x^0 = \mu$, $x^1 = \nu$)

$$\rho_0(\nu) = \alpha\nu + c, \qquad \rho_1(\mu) = \alpha\mu + c \tag{8}$$

with α and c being constants. When $\alpha = 0$, $c = 1$ the original flat metric is retained. For considering Killing tensors different from it α may be set equal to 1 without loss of generality, so that k is of the form (particular Killing tensor)$-c\cdot$(flat metric) throughout this paper. (The minus sign is chosen for later convenience.) The Killing tensor thus constructed from (7) is

$$k^{ik} = \frac{4}{\mu - \nu} \begin{pmatrix} \mu(1-\mu)(c-\nu) & 0 \\ 0 & -\nu(1-\nu)(c-\mu) \end{pmatrix}. \tag{9}$$

2. The Rindler system is conventionally denoted by τ and r,

$$t = r \sinh\tau, \qquad x = r\cosh\tau; \qquad 0 < r < \infty, \quad -\infty < \tau < \infty, \tag{10}$$

with coordinate curves (timelike hyperbolas and spacelike straight lines) given by

$$x^2 - t^2 = r^2, \qquad t/x = \tanh\tau, \tag{11}$$

the metric

$$\mathrm{d}s^2 = r^2\,\mathrm{d}\tau^2 - \mathrm{d}r^2, \tag{12}$$

and the associated Killing tensor

$$k^{ik} = \begin{pmatrix} 1 - \frac{c}{r^2} & 0 \\ 0 & c \end{pmatrix}. \tag{13}$$

3. Killing tensors as metrics, curvature singularities.

¿From the two tensor fields g^{ik} and k^{ik} commuting in the sense of (2) metrics can be generated in different ways. First, by inversion of g^{ik} the covariant flat metric g_{ik} with a Killing tensor $k_{ik} := g_{ij} g_{kl} k^{kl}$ is obtained, or, by inversion of k^{ik}, the covariant metric h_{ik} with a Killing tensor $f_{ik} := h_{ij} h_{kl} g^{jl}$. Then, in a second step, also k_{ik} and f_{ik} may be taken as metrics. In the following h and k will be investigated.

1. In the elliptic and the first hyperbolic system the Killing metrics k and h are

$$k_{ik} = \frac{\mu - \nu}{4} \begin{pmatrix} \frac{c-\nu}{\mu(1-\mu)} & 0 \\ 0 & -\frac{c-\mu}{\nu(1-\nu)} \end{pmatrix} \quad \text{and} \quad h_{ik} = \frac{\mu-\nu}{4} \begin{pmatrix} \frac{1}{\mu(1-\mu)(c-\nu)} & 0 \\ 0 & \frac{-1}{\nu(1-\nu)(c-\mu)} \end{pmatrix} \tag{14}$$

oth of them provide curved manifolds covered by μ and ν. Their determinants change gn at $\mu = c$ or $\nu = c$. If $0 < c < 1$ this occurs in the elliptic system, otherwise the hyperbolic one. In the elliptic case the parameter value c denotes one certain ipse among the coordinate curves with both μ and ν greater or less than c outside d $\nu < c < \mu$ inside. So the exterior has a Lorentzian metric, whereas the interior has positive definite one. For $c < 0$ or $c > 1$ the change of signature occurs on one of the ordinate hyperbolas, which divides analogously the manifold into a concave subset th a Lorentzian metric and a convex one with a definite metric.

An important physical quantity of the obtained curved manifolds with indefinite etrics is the scalar curvature, which for k_{ik} is given by

$$R[k] = 2\,\frac{2\mu\nu - 2c(\mu+\nu) + 3c^2 - c}{(\mu-c)^2\,(\nu-c)^2}. \tag{15}$$

becomes singular exactly at the same parameter value c where the metric changes its gnature, in the hyperbolic cases it vanishes asymptotically for large μs and νs. The rvature $R[h]$ is a little more complicated but has the same singularity structure.

2. The Killing metrics associated with the Rindler coordinate system, which are ngular for $c = 0$, are the following:

$$k_{ik} = \begin{pmatrix} r^4 - cr^2 & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad h_{ik} = \begin{pmatrix} \frac{r^2}{r^2-c} & 0 \\ 0 & \frac{1}{c} \end{pmatrix}. \tag{16}$$

or $c > 0$ there is a domain with a Lorentzian metric between the singular hyperbola $= \sqrt{c}$ and the horizons given by the asymptotes $t = \pm x$, and one with a definite etric on the other side of the singularity. The curvature is given by

$$R[k] = 2\,\frac{2r^2 - 3c}{c\,(r^2-c)^2}, \qquad R[h] = \frac{6\,c^2}{(r^2-c)^2}. \tag{17}$$

4. Extensions, null geodesics

he parts of the domains of separable coordinates, where the curved Killing metric Lorentzian, are regular at the coordinate horizons. For an investigation of possible tensions of these manifolds a transformation from separable coordinates back to t nd x according to (5) and (10) is suitable. Doing so an interesting observation can be ade: The null geodesics with respect to the Killing metrics k_{ik} become straight lines, amely the tangents to the singularities; on the coordinate horizons they coincide with e lightlike directions of the flat metric [2].

In the elliptic and the associated hyperbolic systems now the expressions for the illing metric k and the corresponding curvature are given by

$$k_{ik} = \begin{pmatrix} 1 - x^2 - c & tx \\ tx & -t^2 + c \end{pmatrix} \tag{18}$$

nd

$$R[k] = 2\,\frac{2(1-c)t^2 + 2cx^2 - 3c(1-c)}{[(1-c)t^2 + cx^2 - c(1-c)]^2}. \tag{19}$$

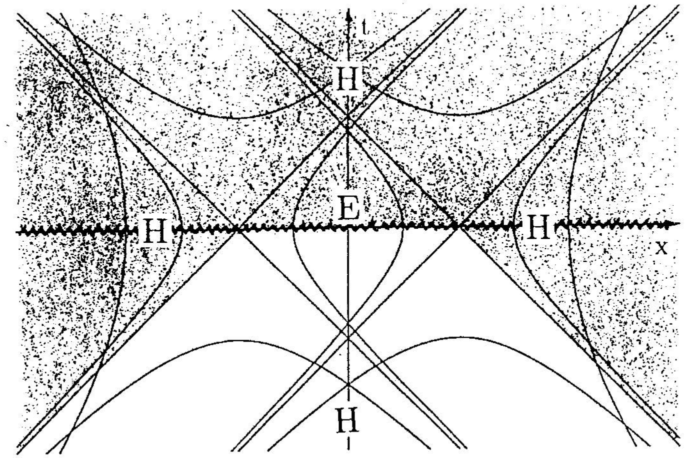

Fig. 1: Singularity, horizons and null geodesics for h_{ik} in the elliptic-hyperbolic case. The domains of elliptic and hyperbolic coordinates are denoted by E and H, respectively.

R is a function only of the determinant of k, the square of which is the denominator, this fact indicates a symmetry. Indeed, by a further transformation of variables,

$$\frac{t^2}{c} + \frac{x^2}{1-c} = \left[\frac{\tau}{\sqrt{c(1-c)}} + 1\right]^2, \qquad \frac{x}{t} = \sqrt{\frac{1-c}{c}}\,\tan\chi, \tag{20}$$

the metric is transformed to the explicit Friedmann-Robertson-Walker form

$$ds^2 = d\tau^2 - \left(\tau + \sqrt{c(1-c)}\right)^2 \left[\frac{\left(\tau + \sqrt{c(1-c)}\right)^2}{c\,(1-c)} - 1\right] d\chi^2. \tag{21}$$

The expression for the metric h in t and x is

$$h_{ik} = -[(1-c)\,t^2 + c\,x^2 - c(1-c)]^{-1} \begin{pmatrix} c - t^2 & tx \\ tx & 1 - c - x^2 \end{pmatrix}, \tag{22}$$

its null vectors are given by

$$n^i_{(1,2)} = \left(tx \pm \sqrt{(1-c)\,t^2 + c\,x^2 - c(1-c)},\; t^2 - c\right). \tag{23}$$

Here the coincidence with the null geodesics of flat space on the horizons is even more distinguished: the integral curves of n^i are straight lines only on the horizons. At $x = t + 1$, for example, $n^i_{(2)} = (t^2 - c)(1, 1)$. This metric is less symmetric than k_{ik}, it does not admit Killing vector fields, as the integrability conditions [7] are not fulfilled, so the Killing tensor f_{ik} (respectively g^{ik}) is a non-trivial one in the sense that it is not a tensor product of Killing vectors.

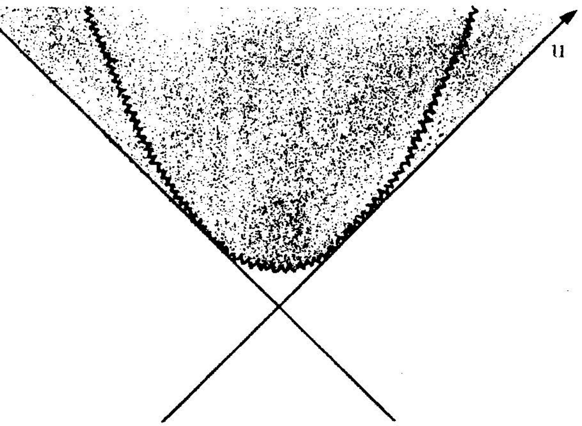

Fig. 2: Causally consistent portion of $\mathcal{M}_2$ in rectangular light coordinates.

For a space-time interpretation the simpler case $c = 0$ is considered, where the null
tors become

$$n^i = (x \pm 1, t). \tag{24}$$

e integral curves are hyperbolas and the curvature singularity lies along $t = 0$ (fig.
$c = 0$ denotes the intermediate case between singular ellipses in the square between
$= -1$ and $x = 1$ and singular hyperbolas in the right and the left wedge.
By a transformation to light coordinates u and v causal relations may be made
nsparent. For

$$t = -\frac{\sqrt{(u-v)^2 - 2\sqrt{2}(u+v) + 2}}{\sqrt{2}}, \qquad x = \frac{u-v}{\sqrt{2}} \tag{25}$$

h

$$ds^2 = \frac{4\,du\,dv}{(u-v)^2 - 2\sqrt{2}(u+v) + 2} \tag{26}$$

unshaded domain in fig. 1 is mapped to the corresponding one in fig. 2, which shows
easonable, although geodesically incomplete space. Analytic continuation across the
izon maps the entire lower half-plane of fig. 1 and yields a timelike continuation of
singularity.
Transformation of the Rindler-related coordinates to t and x results in

$$k_{ik} = \begin{pmatrix} x^2 - c & -tx \\ -tx & t^2 + c \end{pmatrix}, \tag{27}$$

n its null geodesics are the tangents to the singularity, coinciding on the horizon
the null lines of flat space.

For the other Killing metric,

$$h_{ik} = \frac{1}{c\,(t^2 - x^2 - c)} \begin{pmatrix} t^2 - c & -tx \\ -tx & x^2 + c \end{pmatrix}, \tag{28}$$

the null vector fields are given by

$$n^i = \left(x^2 + c,\; tx \pm \sqrt{c\,(x^2 - t^2 + c)}\right). \tag{29}$$

Only on the horizon one of the integral curves is a straight line, the horizon is again generated by the common null geodesics of the two Killing metrics and the flat one. The qualitative features of h_{ik} and k_{ik} are analogous.

Concerning the singularities, for a positive c R (17) is regular everywhere in the domain of the coordinates r and τ, where τ is timelike and the Killing metrics are static. Clearly staticity is inherited from the time symmetry of the Rindler coordinates. In the extended manifold the singularity appears at $t^2 - x^2 = c$, where the determinants of k_{ik} and h_{ik} vanish. This is a spacelike hyperbola, the situation is analogous to the Kruskal extension of the Schwarzschild metric - with the difference that these two-dimensional versions cannot be vacuum solutions.

For a negative c the singularity $x^2 - t^2 = -c$ is timelike, but still has a horizon at $t = \pm x$. If two copies of the manifolds covered by r and τ, bounded by the horizon and the singular hyperbola, are glued together at the singularity and extended smoothly beyond the horizon, a manifold with a singularity and two horizons is obtained. This is exactly a "two-dimensional black hole solution" with an extended source which is described in [8]. The source strength is conserved in virtue of the static nature of the metric.

5. Summary

For the remaining six curvilinear separable coordinate systems the spacetime models generated by Killing metrics are qualitatively the same: Depending on the value of c one obtains always either regular curved manifolds or manifolds bounded by curvature singularities along a certain coordinate line.

For the Killing metrics k_{ik} and h_{ik} considered in this paper the curvature singularities have horizons coinciding with the horizons of the domain of separable coordinates in flat space. In every case these horizons are the only common null geodesics of g_{ik} and both the Killing metrics. The k_{ik}s have two additional properties: They allow for a Killing vector, and written in Minkowski-like coordinates, the null geodesics appear as straight lines. The h_{ik}s have non-trivial (except the Rindler case) Killing tensors of order 2.

Concerning quantum theory one may consider the classical constants of motion $K := k^{ik} p_i p_k$ in the extended phase space, the cotangent bundle of $\mathcal{M}_2$, constructed from the Killing tensors with cotangent vectors to the spacetime manifold. Written symmetrically in configuration and momentum variables, for which canonical commutation relations are imposed (in extended phase space also t and the energy E are conjugate variables), in every case of separable coordinates the quantum versions of K commute with the geodesic hamiltonian, $g^{ik}p_ip_k$. So common eigenstates might be suitable pre-quantum states in a geometric quantization procedure associated with the spacetime domains of separable coordinates and comoving observers. Such a quantization could be attempted parallely and in interaction with one based on mode decompositions in the flat and curved spacetime domains described in this article.

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