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EVOLUTION OF AN ARRAY OF ELEMENTS WITH LOGISTIC TRANSITION PROBABILITY

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The paper addresses the problem how the state of an array of elements changes if the transition probabilities of its elements is chosen in the form of a logistic map. This problem leads to a special type of a discrete-time Markov chain which we simulated numerically for the different transition probabilities and the number of elements in the array. We show that the time evolution of the array exhibits a wide scale of behaviour depending on the value of the total number of its elements and on the logistic constant a. We point out that this problem can be applied for description of a spin system with a certain type of mean field and of the multispecies ecosystems with an internal noise.

1. Introduction

In the last decade there has been a great deal of interest in studies of dynamic systems in presence of noise and fluctuation. The majority of these studies is devoted to the investigation of the effect of noise and fluctuations in a 1D quadratic maps described by a logistic map. For review see, e.g. [1] and [2]. In these studies there were found similar features of the effect of additive and parametric noise in these dynamical systems, a fact which has a great importance and a wide range of applications in physics and in theoretical biology. Another type of investigation regards the behaviour of a network of chaotic elements coupled with local or global maps between the element variables of the form

$$x_{n+1}(i) = (1-\varepsilon)f(x_n(i)) + \frac{\varepsilon}{N}\sum_{j=1}^{N} f(x_n(j)),$$

where x denotes an element variable, $f(x)$ is a certain function of x, n is the number

Nomenclature

- L total number of elements in the array
- S number of elements in the state s_0
- Z number of elements in the state s_1
- p transition probability of changing the state of an element from s_0 to s_1
- q transition probability of changing the state of an element from s_1 to s_0
- a logistic constant

Abbreviations

- MCh Markov chain
- SA stochastical array
- CG center of gravity of a single trajectory
- LJ length of jump in a single trajectory

of a discrete time step, and the elements are indexed by i. Kanenko chose for the function $f(x)$ the logistic map as a model of a system of coupled chaotic elements with a variable interaction [3]. He showed the clustering and coding of attractors in them which are organized so that their change exhibits bifurcation-like phenomena similar to attractors of spin-glasses. His model represents an extension of coupled map lattices (CML) serving as prototypes for spatiotemporal chaos [4]. In all these models their evolution is described by a certain type of interactions of the element variables.

The question which we address in this paper is the description of time evolution of an 1D array of elements which can occur only in two possible states s_0 and s_1. These states are changing in the discrete time with the transition probability given in the form of a logistic map. Defining $r \equiv S/L$, where L and S is the total number of elements in the array and the total number of elements occurring in the state s_0, respectively, we assume that the transition probability of an element to change its state from s_0 to s_1 in a time step is

$$p = a\,r(1-r), \tag{1}$$

where a is the logistic constant. Thus, we linked the transition probability of an element state with the number of array elements occurring in a the state s_0. We show the time evolution of such a 1D array (SA for short) is a stochastic process, described by a Markov chain, which exhibits a rich behaviour depending on the value of the logistic constant a and total number of elements L.

Since we do not take into account the memory, the stochastic process of such system is to a discrete-time Markov chain (MCh) [6] determined by the transition probability between next time steps. In what follows we determine analytically this transition probability and present the results of a numerical simulation of the MCh for different value of a and L in the form of single trajectories and histograms. Finally, we discuss the potential applicability of these results in physics and mathematical biology.

2. The Markov chain

We consider an array of L elements each of which can occur in one of two possible states s_0 and s_1. Their states change stochastically in discrete time. If S is the number of elements occurring in the state s_0 (s_0 – elements) and Z is the number of those occurring in the state s_1 (s_1 – elements) we have $L = S + Z$. We assume that the transition probability of changing the state of an element from s_0 to s_1 in a time step depends on the total number of s_0-elements in the array via the logical map (1). If the transition probability from s_0 to s_1 is denoted by p and from s_1 to s_0 by q, then we assume further that $p + q = 1$.

The stochastic variable, which describes the time evolution of our system, is S. The considered stochastic evolution of the array represents a Markov chain determined in a discrete time and characterized by the transition probability $P(S_i|S_{i+1})$ where S_i is the number of s_0-elements in the time t_i and S_{i+1} is that in the time t_{i+1}. This transition probability consists of a "kinetic" and a "dynamic" parts. The kinetic part is given by the number of ways in which our array with S_i s_0-elements in time t_i passes over to S_{i+1} s_0-elements in time t_{i+1}. Using combinatorial arguments we get for the partial probability of the transition $P_l(S_i|S_{i+1})$ from S_i to S_{i+1} on the path with $(S_i - l)$ $s_0 \to s_1$ transitions and $(S_{i+1} - l)$ $s_1 \to s_0$ transitions the following expression

$$P_l(S_i|S_{i+1}) = \binom{s_i}{l}(1-p)^l p^{S_i-l}\binom{L-S_i}{S_{i+1}-l}(1-q)^{L-S_i-S_{i+1}+l} q^{S_{i+1}-l}$$

Summing over all possible paths, we get for the total transition probability

$$P(S_i|S_{i+1}) = \sum_{l=\max(0,S_i+S_{i+1}-L)}^{\min(S_i,S_{i+1})} P(S_i|S_{i+1}). \tag{2}$$

The formula (2) for the transition probability is quite general, therefore, we have to specify in it the transition probabilities p and q. As said in Introduction, we take for the probability p the logistic map in the form (1). This is why the logistic map has very rich dynamics [5] depending on the values of the logistic constant a. If $3 > a \geq 1$ the fixed point at $x = 1 - 1/a$ is an attractor. At $a = 3$ the logistic map bifurcates and as a increases the successive bifurcations give rise to a cascade of period doublings. In the range $4 > a > 3.570$ the map exhibits a chaotic behaviour. The choice for this transition probability p was motivated mainly by the fact that this map is mathematically well understood in the whole range of the logistic constant a. Since the Markov chain which we consider is mathematically rather complicated we will, in what follows, make only its numerical simulation for the relevant values of the total number of the array elements L and logical constant a.

3. Computer simulation of the Markov chain

For the description of the evolution of SA by means of MCh we need two input parameters: (i) the total number of elements L and (ii) the value of logistic constant a. We determine some relevant single trajectories of our MCh and the corresponding histograms for $a = 2.95, 3.1$ and 3.5 as well as for $L = 200, 400$ and 600. The single trajectory of MCh over 500 time steps after the initial transient phase for $a = 2.95$ and $L = 200$ is shown in Fig.1. Here the full line depicts the value of the attractor of the corresponding logistic map. The value of S_n/L for odd and even time steps are denoted by (+) and (*), respectively. The stars and crosses are connected separately by lines as

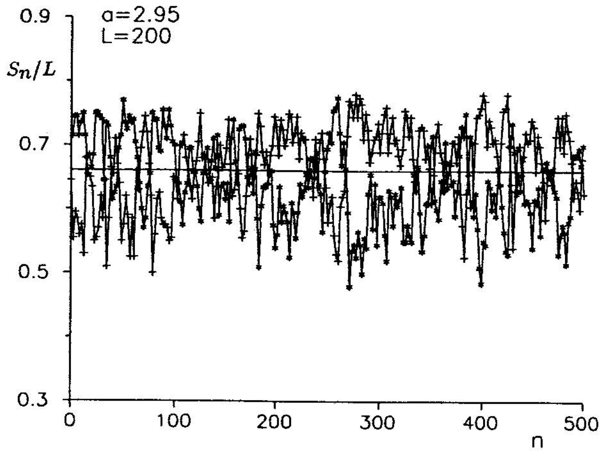

Fig. 1. Single trajectory over 500 time steps for $a = 2.95$ and $L = 200$. The solid line parallel to the time axes denotes the value of the fixed point for the corresponding logistic map.

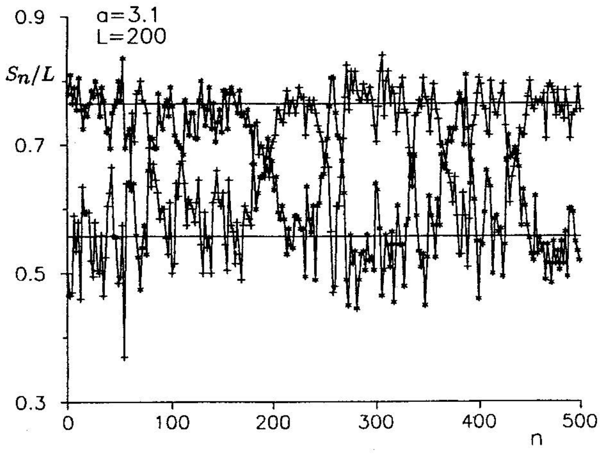

Fig. 2. Single trajectory over 500 time steps for $a = 3.1$ and $L = 200$. The solid lines parallel to the time axes represent the periodic points.

a guide for the eye. Fig. 1 shows that at $a = 2.95$ stars and crosses are placed at both sides of the full line. Despite the fact that a is less than the critical value for bifurcation of the logistic map, the values of S_n/L are not distributed in a totally random way, but they form small areas where the crosses and stars are separated, i.e. where MCh exhibits a bifurcation-like behaviour. This fact has been observed also by many other authors when studying dynamical systems in presence of noise, see, e.g. [7]. The effect of the number of elements in SA on the single trajectory just above the bifurcation is

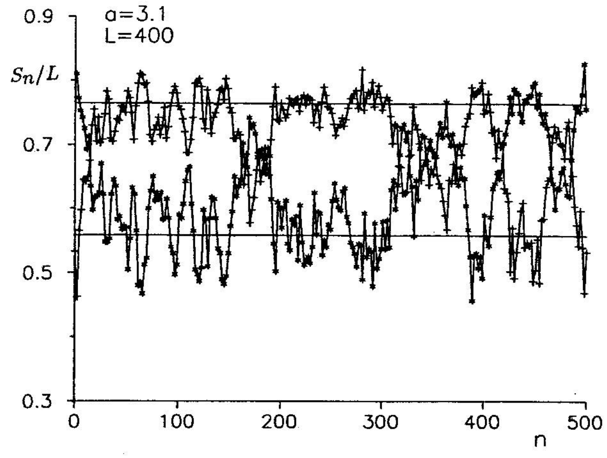

Fig. 3. Single trajectory over 500 time steps for $a = 3.1$ and $L = 400$.

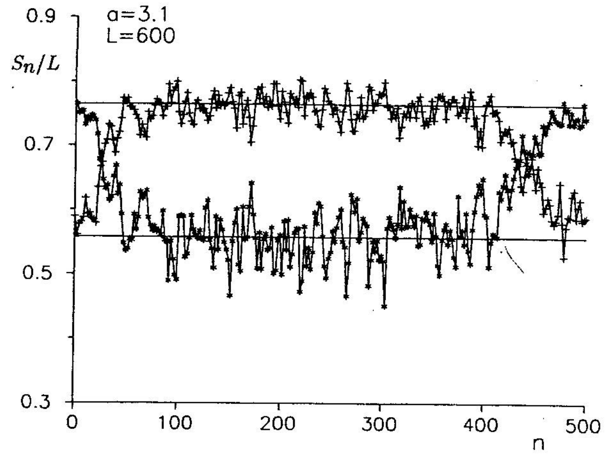

Fig. 4. Single trajectory over 500 time steps for $a = 3.1$ and $L = 600$.

shown in Figs 2, 3, 4 which clearly indicates that the larger is L the less is the dispersion of the value of S_i/L around the periodic points of corresponding logistic map. For the statistical analysis of our MCh we take the following statistical data as the most relevant for the evaluation of the behaviour SA:
(i) the value of the random variable S;
(ii) the "center of gravity" (CG) of S defined as $(S_i - S_{i+1})/2$ and
(iii) the product of differences of neighbouring values of S related to the absolute value

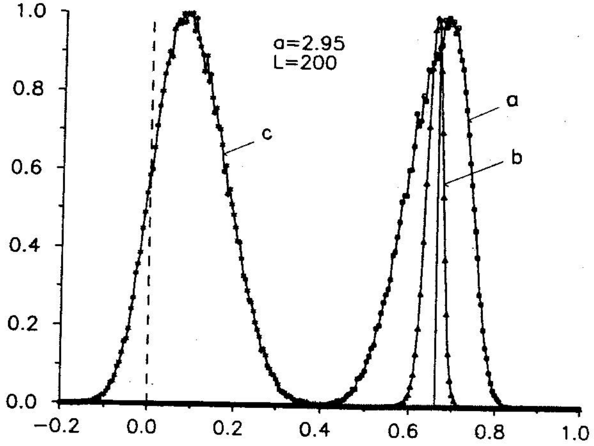

Fig. 5. Histograms of some relevant statistical quantities. The lines are guide for the eye. The histogram of S, CG and LJ is depicted by curves a,b, c, respectively. The x-coordinate denotes the values of S, SG, and LJ. The positive value of LJ has the jump in the "right" direction otherwise it has negative value. The histograms here are for $a = 2.95$ and $L = 200$.

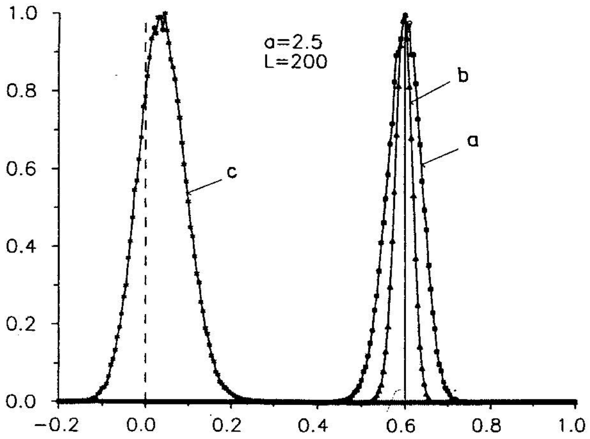

Fig. 6. Histograms of S, CG and LJ for $a = 3.5$ and $L = 200$.

of one of them $-(S_{n+1} - S_n)(S_n - S_{n-1})/|S_n - S_{n-1}|$, which represents the length of jumps (LJ) with the sign denoting if the jump is in the "right" direction i.e. opposite to the direction of the jump in the previous step. The most frequent events in the evolution of the MCh is that the successive jumps have opposite direction.

All these statistical quantities are displayed in figures in the form of normed histograms in which any value of a certain statistical quantity is related to its maximum value occurring in a single trajectory. These histograms are shown for $L = 200$ and

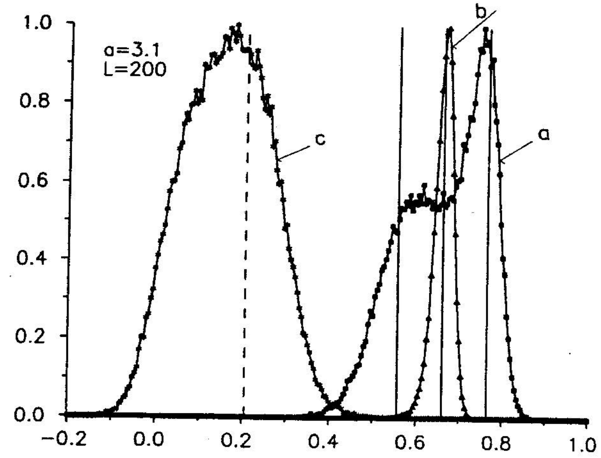

Fig. 7 Histograms of S, CG and LJ for $a = 3.1$ and $L = 200$.

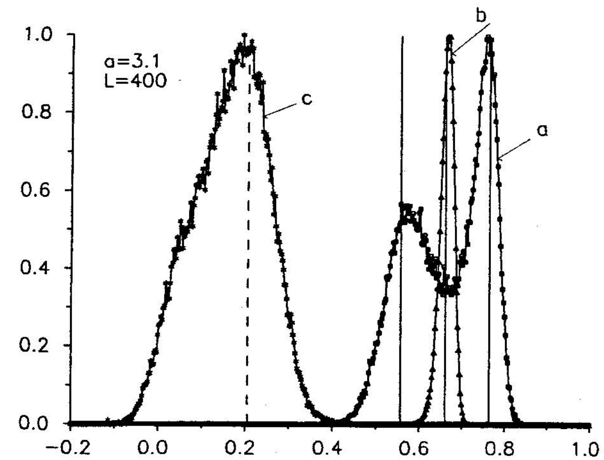

Fig. 8. Histograms of S, CG and LJ for $a = 3.1$ and $L = 400$.

$a = 2.5$ and 2.95 in Figs 6 and 5, respectively. They show that the probability distribution given by the histogram of S (denoted as histogram a) is symmetric but slightly shifted from the solid line and the histogram of CG (histogram b) has a slightly asymmetrical form which is not fully smooth. The histogram (c) for LJ have a complex structure. Both histograms (b) and (c) are broader for $a = 3.1$ than for 2.5. The effect of the number of elements L on the form of histograms is demonstrated in Figs 7, 8, 9 which are slightly above the bifurcation point of the logistic map. For small number of

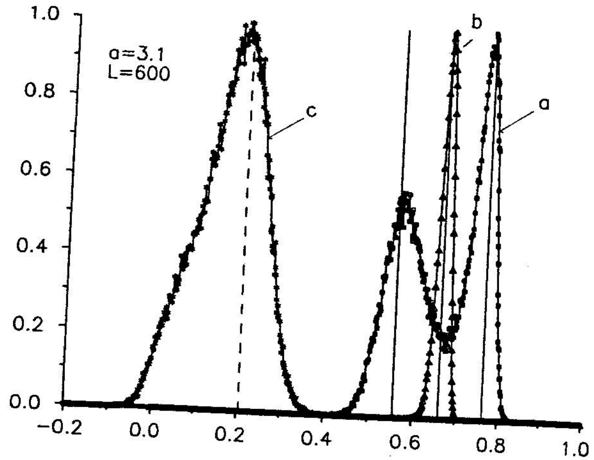

Fig. 9. Histograms of S, CG and LJ for $a = 3.1$ and $L = 600$.

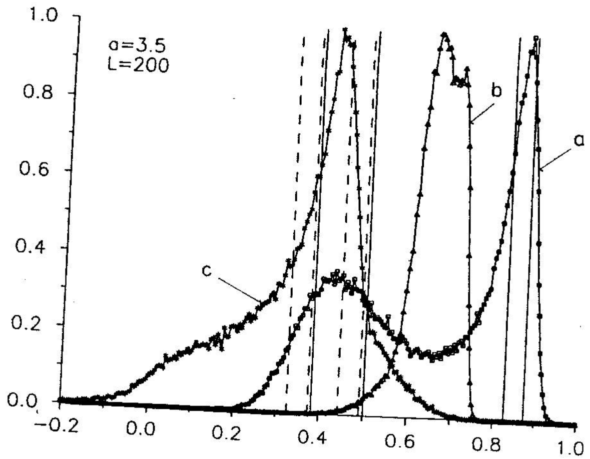

Fig. 10. Histogram of S, CG and LJ for $a = 3.56$ and $L = 200$. The dotted lines parallel to the abscises denote the difference between the periodic points.

elements ($L = 200$) the values of the odd steps are scattered around one of the solid lines and those of even steps are scattered about the other one. After relatively short interval they exchange their positions. The larger values of elements L the narrower are histograms of S and they are longer stuck to one of the solid lines. The maximum values of the histograms of S appear between the element lines. The histogram CG is approximately centered around the center of the solid lines. Both histograms of S and LJ are asymmetric and broader at the lower side. For $L = 200$ the left maximum

has a form of a shoulder rather than of a peak There is a pronounced tail to negative values in the histogram of LJ which reflects the presence of numerous regions where crosses and stars interchange their positions in the single trajectory. With increasing number of L all the histograms become narrower. As an example for the simulation of the behaviour of SA nearer to the chaotic regime of logical map we have taken $a = 3.5$. Fig. 10 shows that the histograms (a) and (b) lay farther from the two solid lines than those for $a = 3.1$. For $a = 3.5$ the logistic map has four periodic points and instead of four peaks in histogram CG we have only one rather narrow peak with long tail for the small values of LJ.

4. Conclusions and possible applications

From what has been said so far it follows that
(i) for large values of L the histograms of CG and LJ become narrower and the peaks in the neighbourhood of the full lines become sharper;
(ii) the histograms for LJ are sharper and smoother;
(iii) single trajectory shows a bifurcation-like behaviour slightly before the bifurcation point;
(iv) increase of the value of a for constant L causes broadening of all the histograms.
Since the time evolution of considered SA described by the MCh is rather general it can simulate a large class of complex systems in physics, biology and in general system theory. Let us briefly mention only two typical examples, namely the linear Ising chain with a certain type of mean field and a population with noise. The linear chain of spins consists of an array of spins having two different directions [8]. If the magnetic field produced by the spins is a linear function of the spin direction and the probability of change of the spin state is a quadratic function of the mean field then such a linear spin array can be described by our MCh. In the theoretical biology the simplest model for the population dynamics is described by the logistic equation [9]. This model simulates the behaviour of single population having discrete non-overlapping generation. However such a population is generally exposed to fluctuations, environmental noise and other change events. The common way to investigate such a population is to take some logistic equation for its description and then to add to it some sorts of noise [10]. We find it more appropriate if one considers a priori a stochastical evolution of a population in the form of our Markov chain, where S_i is the magnitude of population in the time t_i and then determines under which condition it can be approximated by the logistic equation. As we have demonstrated above our SA described by MCh approaches to a deterministic logistic map when L is very large and it exhibits more stochasticity when L is small. This corresponds also to the common feeling that a small population is more sensitive for stochastical effects than a larger one.

The considered MCh could be the adequate model for describing the real population dynamics of multispecies ecosystems. The pure single species ecosystems , described by the deterministic logistic equation, occur in nature only rarely. If we consider realistic multispecies ecosystems we necessarily have to take into account the interspecies interaction and the environmental noise. This can be done by considering an array of elements obeying the logistic equation and simulating the possible interspecies interaction and environmental noise.

These two examples demonstrate only the fact that the considered SA may have several possible applications in different areas of science. The simulation of a multi-species ecosystem in the whole range of its realistic parameters will be the subject of a subsequent paper.

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