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CORRELATIONS, CORRELATION INTEGRALS AND APPLICATION TO BOSE-EINSTEIN INTERFEROMETRY[1]

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We review the basic notion of correlations in point processes, adapted to the language of high energy physicists. The measurement of accessible information on correlations by means of correlation integrals is summarized. Applications to the measurement of Bose-Einstein interferometry as well as some pitfalls are discussed.

1. Introduction

In moving to ever higher energies in particle physics, the experimentalist faces a rapidily increasing complexity of the observed events. Beyond posing questions of physics alone, the task to compare experimental data with competing theoretical and phenomenological models raises many questions purely *statistical* in nature. The latter define a field of their own: *multiparticle statistics.*

The task of multiparticle statistics is to provide a framework for maximal utilization of the information provided by experiments. This includes questions of measurement of single and multiparticle spectra, correlations among two and more particles, statistical errors due to finite event-sample size, limited detector acceptance, misidentification of track and much more.

In these lectures, we summarize basic concepts of point processes, correlations and their measurement. To demonstrate the discriminative power of advanced correlation measurements, we discuss a recent test of dynamical assumptions in modeling Bose-Einstein correlations.

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2. Cross sections, point processes and correlations

Correlations (in the widest sense) among final state particles are commonly expressed in terms of exclusive or inclusive differential cross sections denoted $\sigma_{\rm excl}$ and $\sigma_{\rm incl}$ respectively. To be specific, we recall some common definitions [1]:

$$j_N(p_1,\ldots,p_N) \equiv \frac{1}{\sigma_{\rm tot}} \frac{d^{3N}\sigma_{\rm excl}}{d^3p_1 d^3p_2 \ldots d^3p_N}, \quad (1)$$

$$\rho_q(p_1,\ldots,p_q) \equiv \frac{1}{\sigma_{\rm tot}} \frac{d^{3q}\sigma_{\rm incl}}{d^3p_1 d^3p_2 \ldots d^3p_q}, \quad (2)$$

p_i is the three-momentum of the i-th particle, and $\sigma_{\rm tot} = \sum_N \sigma_N$ is the total (inelastic) cross section, with σ_N the integrated cross section for events with N final state particles (here to be taken as identical for simplicity of notation). The ratio $P_N = \sigma_N/\sigma_{\rm tot}$ gives the multiplicity distribution in full phase-space.

The densities j_N and ρ_q are *very* different objects: $j_N(p_1,\ldots,p_N)d^3p_1\ldots d^3p_N$ is proportional to the probability that in an event with exactly N particles we find simultaneously the first particle in a box of size d^3p_1 centered in p_1, the second in a box at $p_2,\ldots$, and the N-th at p_N. Thus, j_N characterizes the statistical properties of samples built *exclusively* from events with exactly N particles (exclusive samples). By contrast, $\rho_q(p_1,\ldots,p_q)d^3p_1\ldots d^3p_q$ is the *average number* of unordered q-tuples of particles with momenta simultaneously within infinitesimal boxes centered on $p_1,\ldots,p_q$ per event (no matter what N is). Therefore ρ_p characterizes samples which include *all* events (inclusive samples). While various other names are in use, we follow Ref. [2] in referring to j_N as a *Janossy density* and to ρ_q as a *factorial moment density*.

Both functions are symmetrized with respect to permutations of their arguments. In fact, the labels "first" particle, "second" particle, ... are arbitrary and carry no physical information. Therefore it is customary to count all $N!(q!)$ permutations of labels as separate, independent events (q-tuples). This symmetrization has nothing to do with quantum mechanical "indistinguishability" of particles, but reflects a mere convention designed to simplify the resulting formalism.

While $j_N/N!P_N$, normalized to unity, represents a conventional *joint probability density* of N random variables p_i, it is a mistake to regard ρ_q as such. It is constructed from inclusive samples, where the number of random variables (N) is a again a random variable.

In fact, integration over an arbitray region of phase-space Ω gives the q-th factorial moment of the random multiplicity n in this domain:

$$\xi_q(\Omega) = \int_\Omega \rho_p(p_1,\ldots,p_q)d^3p_1\ldots d^3p_q = \langle n^{[q]}\rangle_\Omega. \quad (3)$$

We use the common notation $n^{[q]} \equiv n(n-1)\cdots(n-q+1)$ for factorials.

All the above definitions are given in terms of cross sections. However, all *experimental* multiplicity and correlation measurements can be viewed as particular *counting procedures* of particles or particle-tuples in certain domains. Thus, we pose the questions: How do we have to count in order to obtain a certain type of information and, moreover, how do we count most efficiently?

The appropriate tool to tackle such questions is the theory of point processes [3, 2]. Consider a sample of N_{ev} events, each labeled by an index $a = 1,\ldots,N_{ev}$. For greater generality, we subsequently denote the "positions" of the particles (=points) by $\mathbf{X}_i^a$, ($i = 1,\ldots,N$), where $\mathbf{X}$ can refer to *any* set of coordinates. Frequently used examples are rapidity y, rapidity-azimuth (y,Φ) or even a combination of continuous and discrete variables such as (p,s), with s labeling the charge, spin or species of the particle[3]

The density of such points at $\mathbf{x}$ in one particular event a is most conveniently represented by the "random Dirac comb"

$$\hat\rho_1^a(\mathbf{x}) = \sum_{i_1=1}^{N} \delta(\mathbf{x} - \mathbf{X}_{i_1}^a), \quad (4)$$

which, when integrated over a certain domain Ω, just gives the number of particles n in that domain. More generally, the simultaneous behavior (=correlation) of q of these particles in that event is represented by the restricted tensor product of Dirac combs

$$\hat\rho_q^a(\mathbf{x}_1,\ldots,\mathbf{x}_q) = \sum_{i_1\neq i_2\neq\ldots\neq i_q=1}^{N} \delta(\mathbf{x}_1 - \mathbf{X}_{i_1}^a)\delta(\mathbf{x}_2 - \mathbf{X}_{i_2}^a)\ldots\delta(\mathbf{x}_q - \mathbf{X}_{i_q}^a), \quad (5)$$

which just acts as a q-tuple counter in event a. Note that it doesn't make sense to count the same particle more than once (hence the restriction of the indices in the multiple sum).

Meaningful results are extracted by averaging over the inclusive event sample, to yield the q-tuple density

$$\rho_q = \langle\hat\rho_q\rangle = N_{ev}^{-1} \sum_{a=1}^{N_{ev}} \hat\rho_q^a, \quad (6)$$

which is nothing but the counting prescription for the factorial moment densities eq. (2).

A point process is fully determined by the knowledge of either all ρ_q, all j_N or, most conveniently, by its generating functional. The latter is defined by

$$Z[\lambda(\mathbf{x})] = \left\langle \exp\left(\int \hat\rho(\mathbf{x}) \log[1+\lambda(\mathbf{x})]d\mathbf{x}\right)\right\rangle \quad (7)$$

$$= \sum_{q\geq 0} \frac{1}{q!} \int \rho_q(\mathbf{x}_1,\ldots,\mathbf{x}_q)\lambda(\mathbf{x}_1)\cdots\lambda(\mathbf{x}_q)d\mathbf{x}_1\cdots d\mathbf{x}_q. \quad (8)$$

Once we know $Z[\lambda(\mathbf{x})]$, the ρ_q as well as the j_N can be obtained via functional derivatives with respect to the test function λ:

$$\rho_q(\mathbf{x}_1,\ldots,\mathbf{x}_q) = \left.\frac{\delta^q Z[\lambda(\mathbf{x})]}{\delta\lambda(\mathbf{x}_1)\cdots\delta\lambda(\mathbf{x}_q)}\right|_{\lambda=0} \quad (9)$$

[3] In this case all following integrations contain implicitly sums over s.

$$j_N(\mathbf{x}_1,\ldots,\mathbf{x}_N) = \left.\frac{\delta^q Z[\lambda(\mathbf{x})-1]}{\delta\lambda(\mathbf{x}_1)\cdots\delta\lambda(\mathbf{x}_N)}\right|_{\lambda=0}. \qquad (10)$$

In order to understand what we mean by "correlations" we first discuss the meaning of *statistical independence* in point processes. The latter is defined by full factorization of the factorial moment densities $\rho_q(\mathbf{x}_1,\ldots,\mathbf{x}_q) = \rho_1(\mathbf{x}_1)\rho_1(\mathbf{x}_2)\cdots\rho_1(\mathbf{x}_q)$ for all q. In this case we can readily sum up the series (8) to obtain

$$Z[\lambda(\mathbf{x})] = \exp\left(\int \rho_1(\mathbf{x})\lambda(\mathbf{x})d\mathbf{x}\right), \qquad (11)$$

which is the generating functional of the so-called Poisson process. In this process, the multiplicity of points in *any* arbitrary domain Ω follows a Poisson distribution. This process plays a similar central role for point processes as the Gaussian does for the statistics of continuous random variables.

Genuine correlations among points are then quantifiable by *deviations* from the Poisson process. Consider the family of functions, called *cumulant densities*, obtained by functional derivatives of log $Z[\lambda]$:

$$C_q(\mathbf{x}_1,\ldots,\mathbf{x}_q) = \left.\frac{\delta^q \log Z[\lambda(\mathbf{x})]}{\delta\lambda(\mathbf{x}_1)\cdots\delta\lambda(\mathbf{x}_q)}\right|_{\lambda=0}. \qquad (12)$$

For the Poisson process (11) we see that $C_1 = \rho_1$ and all higher cumulant densities vanish. In this sense, nonvanishing C_q with $(q \geq 2)$ quantify genuine correlations, i.e. deviations from the Poisson process.

It is quite possible that only a few orders of cumulants are nonzero. For example in heavy ion reactions cumulants of third and higher orders are highly suppressed. Such processes can be modeled as poisson processes of "clusters", each cluster decaying into one or 2 daughter particles.

The generating functional $Z[\lambda]$ is a convenient bookkeeping device for establishing all kinds of relations among j_N, ρ_q and C_q, for example

$$C_2(\mathbf{x}_1,\mathbf{x}_2) = \rho_2(\mathbf{x}_1,\mathbf{x}_2) - \rho_1(\mathbf{x}_1)\rho_1(\mathbf{x}_2), \qquad (13)$$

$$\begin{aligned} C_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \;=\;& \rho_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) - \rho_1(\mathbf{x}_1)\rho_2(\mathbf{x}_2,\mathbf{x}_3) - \rho_1(\mathbf{x}_2)\rho_2(\mathbf{x}_3,\mathbf{x}_1) \\ & -\rho_1(\mathbf{x}_3)\rho_2(\mathbf{x}_1,\mathbf{x}_2) + 2\rho_1(\mathbf{x}_1)\rho_1(\mathbf{x}_2)\rho_1(\mathbf{x}_3) \quad \text{etc.} \end{aligned} \qquad (14)$$

Notice the subtration of various products of lower order factorial moment densities in the construction of cumulants, sometimes called "removal of combinatorial background". It is this property that makes cumulants a favourable tool for discriminative experimental analysis, but at the same time more difficult to measure.

3. Correlation integrals

If we had experimental knowledge of correlations in terms of either j_N, ρ_q or C_q to *all* orders, the multiparticle production process would be *completely* determined. This can never, of course, be achieved in practice. However, while higher-order correlation

functions can never be sampled fully differentially, one can still try to extract as much information as possible from a give data sample by measuring *integrals* over various domains Ω as in eq. (3).

Conventional measurements of correlations proceed first to discretize the continuous variable $\mathbf{X}$ ("binning the data") and then to find ξ_q by averaging over all events the counts $n_m^{[q]}$ in every bin [4],

$$\xi_q^{conv} = \left\langle \sum_{\text{bins } m} n_m^{[q]} \right\rangle. \qquad (15)$$

By contrast, so-called *correlation integrals* rely on distances between pairs of points $\mathbf{X}_{i_1i_2} \equiv |\mathbf{X}_{i_1} - \mathbf{X}_{i_2}|$ rather than counting particles in predefined bins [4, 5]. More generally, every correlation integral of order q assigns "size" to every possible q-tuple of particles. The way this assignment is done distinguishes the different versions of correlation integrals. Finally, they count the number of q-tuples with a given size ϵ ("differential forms) or the ones smaller than a given size ("integral forms"). For a large data sample, this corresponds to an integration over ρ_q in specific domains $\Omega(\epsilon)$.

For the Star integral [6], the domain Ω is given by the collection of N spheres of radius ϵ, each centered at one of the N particles in the event. For a given event a the number of particles ("sphere count") within one of these spheres is, not counting the particle at the center $\mathbf{X}_{i_1}$,

$$a \equiv \hat{n}(\mathbf{X}_{i_1},\epsilon) \equiv \sum_{i_2=1}^{N} \Theta(\epsilon - \mathbf{X}_{i_1i_2}), \quad i_2 \neq i_1, \qquad (16)$$

where a is an "ultra short" notation needed for some lengthy formulae below. With this elementary counter the factorial moment of order q is simply obtained by

$$\xi_q^{\text{Star}}(\epsilon) = \langle \sum_{i_1} \hat{n}(\mathbf{X}_{i_1},\epsilon)^{[q-1]} \rangle = \langle \sum_{i_1} a^{[q-1]} \rangle. \qquad (17)$$

One can show [6] that the above counting prescription corresponds to integrating eq. (3) using for Ω a particular "Star" domain implemented via theta functions $\Theta_{1j} \equiv \Theta(\epsilon - |\mathbf{x}_1 - \mathbf{x}_j|)$, restricting all $q-1$ coordinates $\mathbf{x}_j$ to within a distance ϵ of $\mathbf{x}_1$:

$$\xi_q^{\text{Star}}(\epsilon) = \int \rho_q(\mathbf{x}_1,\ldots,\mathbf{x}_q)\Theta_{12}\Theta_{13}\ldots\Theta_{1q}d\mathbf{x}_1\ldots d\mathbf{x}_q. \qquad (18)$$

The superiority of correlation integrals in general over the conventional Bialas-Peschanski factorial moments [4] is discussed and demonstrated in [5]. The further advantage of the particular "Star" domain lies in the fact that eq. (17) requires typically $N_{ev} * N^2$ computation steps for *any* order q, whereas other types of correlation integral have order $N_{ev} * N^q$ complexity, which quickly becomes unmanageable for higher orders.

In order to eliminate, among other things, the overall total cross section, it has become customary in high energy physics to measure *normalized factorial moments* [4].

The denominator used for such normalization should be made up of the *uncorrelated* background, ρ_1^q. While it can be implemented in a number of ways, we prefer the "vertical" normalization, in which ρ_1^q is integrated over exactly the same domain Ω as the inclusive density ρ_q in the numerator. Thus for the Star integral, the normalized moments is

$$F_q^{\rm Star}(\epsilon) \equiv \frac{\xi_q^{\rm Star}}{\xi_q^{\rm norm}} = \frac{\int \rho_q(\mathbf{x}_1,\ldots,\mathbf{x}_q)\Theta_{12}\Theta_{13}\ldots\Theta_{1q}d\mathbf{x}_1\ldots d\mathbf{x}_q}{\int \rho_1(\mathbf{x}_1)\ldots\rho_1(\mathbf{x}_q)\Theta_{12}\Theta_{13}\ldots\Theta_{1q}d\mathbf{x}_1\ldots d\mathbf{x}_q}. \quad (19)$$

We have shown [6] that the denominator $\xi_q^{\rm norm}$ is given by the following double event average: with $\mathbf{X}_{i_1i_2}^{ab} \equiv |\mathbf{X}_{i_1}^a - \mathbf{X}_{i_2}^b|$ measuring the distance between two particles *taken from different events* a and b, and the "ultra short" notation

$$b \equiv \hat{n}_b(\mathbf{X}_{i_1}^a, \epsilon) = \sum_{i_2} \Theta(\epsilon - \mathbf{X}_{i_1i_2}^{ab}) \quad (20)$$

we get

$$\xi_q^{\rm norm}(\epsilon) \equiv \left\langle \sum_{i_1} \hat{\xi}_q^{\rm norm}(i) \right\rangle = \left\langle \sum_{i_1} \langle b \rangle^{q-1} \right\rangle. \quad (21)$$

Note that the outer event average and sum over i_1 are taken over the center particle taken from event a, each of which is used as the center of sphere counts $\hat{n}_b(\mathbf{X}_{i_1}^a, \epsilon)$ taken over other events $b \neq a$ in the inner event average. We thus see the natural emergence of the heuristic procedure of normalization known as "event mixing" [6, 5]. This counting prescription has complexity $N_{\rm ev}^2 * N^2$ – unmangeable for large event samples. However, in practical applications it is sufficient to reduce the inner average over b-events to a small subsample $b = a-1, a-2, \ldots, a-A$, consisting only of $A-1$ events and requiring only $A * N_{ev} * N^2$ computation steps. This procedure we call "reduced" event mixing.

Another essential advantage of the Star domain is the fact that on top of the F_q^{Star} we get the integrals of cumulants (almost) for free, once the counts a and b per particle i are perfomed. Integrating the C_q over the Star integral domain, we define normalized (factorial) cumulants $K_q^{\rm Star}(\epsilon) \equiv f_q(\epsilon)/\xi_q^{\rm norm}(\epsilon)$, with

$$f_q(\epsilon) \equiv \int C_q(\mathbf{x}_1,\ldots,\mathbf{x}_q)\Theta_{12}\Theta_{13}\ldots\Theta_{1q}d\mathbf{x}_1\ldots d\mathbf{x}_q. \quad (22)$$

The latter can be written entirely in terms of the sphere counts a and b introduced previously! Defining for convenience the "i-particle cumulant" $\hat{f}_q(i)$ so that $\left\langle \sum_i \hat{f}_q(i) \right\rangle = f_q$, we find [6]

$$\hat{f}_2(i) = a - \langle b \rangle, \quad (23)$$

$$\hat{f}_3(i) = a^{[2]} - \langle b^{[2]} \rangle - 2a\langle b \rangle + 2\langle b \rangle^2, \quad \text{etc.} \quad (24)$$

While computationally more expensive than the Star integral by orders of magnitude, other types of integration domains Ω are also useful, in particular when a comparison of theoretical models with data dictate a specific choice of variables and integration

domain. This is typically the case in measurements of Bose-Einstein correlations, where a preferred variable is the 4-momentum difference $q_{ij} = [(\mathbf{p}_i - \mathbf{p}_j)^2 - (E_i - E_j)^2]^{1/2}$.

We then may study differential forms of integrations over ρ_q or C_q in a fully symmetrical manner (called GHP topology) [7] such as

$$C_q(\epsilon) = \int C_q(\mathbf{p}_1,\ldots,\mathbf{p}_q)\delta(\epsilon - \sum_{i<j=1}^{q} q_{ij}^2)d^3\mathbf{p}_1\cdots d^3\mathbf{p}_q. \quad (25)$$

Other q-tuple size prescriptions (used below) are obtained by changing the argument of the δ-function appropriately, e.g. $\delta(\epsilon - \sum_{i<j=1}^{q} q_{ij})$ or $\delta(\epsilon - \max(q_{12},\ldots,q_{q-1,q}))$.

The corresponding counting prescriptions are conveniently written in terms of the generic q-tuple counter

$$I_{i_1i_2\ldots i_q}^{e_1e_2\ldots e_q}(\epsilon) = \begin{cases} 1 & \text{if the "size" of the } q\text{-tuple is within} \epsilon + \delta\epsilon \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

where the q-tuple is composed of particle i_1 taken from event e_1, particle i_2 taken from a different event e_2, etc., and the "size" is evaluated, e.g., according to one of above quoted prescriptions.

The counting algorithms for $\rho_q(\epsilon)$ and the first $C_q(\epsilon)$ are

$$\rho_q(\epsilon)\delta\epsilon = \left\langle \sum_{i_1\neq\cdots\neq i_q} I_{i_1i_2\cdots i_q}^{aa\cdots a}(\epsilon) \right\rangle \quad (27)$$

$$C_2(\epsilon)\delta\epsilon = \left\langle \sum_{i\neq j} I_{ij}^{aa}(\epsilon) \right\rangle_a - \left\langle\left\langle \sum_{i,j} I_{ij}^{ab}(\epsilon) \right\rangle_b\right\rangle_a \quad (28)$$

$$C_3(\epsilon)\delta\epsilon = \left\langle \sum_{i\neq j\neq k} I_{ijk}^{aaa}(\epsilon) \right\rangle_a - 3\left\langle\left\langle \sum_{i\neq j,k} I_{ijk}^{aab}(\epsilon) \right\rangle_b\right\rangle_a + 2\left\langle\left\langle\left\langle \sum_{i,j,k} I_{ijk}^{abc}(\epsilon) \right\rangle_c\right\rangle_b\right\rangle_a \quad (29)$$

Integrals (25) over uncorrelated tensor products of ρ_1, needed for normalization, are sampled in similar ways [7]:

$$\rho_1 \otimes \rho_1 \otimes \cdots \otimes \rho_1(\epsilon)\delta\epsilon = \left\langle\left\langle \cdots \left\langle \sum_{i_1,i_2,\cdots,i_q} I_{i_1i_2\cdots i_q}^{e_1e_2\cdots e_q}(\epsilon) \right\rangle_{e_1} \right\rangle_{e_2} \cdots \right\rangle_{e_q}. \quad (30)$$

Note that multiple event averages, e.g. $\langle\langle\ldots\rangle_a\rangle_b = \sum_{a\neq b}/N_{\rm ev}(N_{\rm ev}-1)$, always run over unequal events to avoid noticeable sampling biases. Again, inner event averages can run over a small fraction of the full sample to keep computing times in manageable ranges [6].

4. Applications to Bose-Einstein correlation measurements

As an application, we discuss a recent test [8] of quantum statistical (QS) models for Bose-Einstein correlations among identical pions [9, 10, 11]. The latter postulate a specific form of the generating functional (7) with C_2 as freely parametrizable function. Once C_2 is determined from experiment, all higher cumulant densities are fully specified with no further adjustable parameter. One therefore can test experimentally the validity of the postulated generating functional by comparing higher order cumulants from experiment with the model predictions, without having to rely on a particular parametrization of C_2.

When relative phases are neglected, the second and third (reduced) "QS cumulants" of interest are [11]

$$k_2 \equiv \frac{C_2}{\rho_1 \otimes \rho_1} = 2p(1-p)d_{12} + p^2 d_{12}^2, \quad (31)$$

$$k_3 \equiv \frac{C_3}{\rho_1 \otimes \rho_1 \otimes \rho_1} = 2p^2(1-p)[d_{12}d_{23} + d_{23}d_{31} + d_{31}d_{12}] + 2p^3 d_{12}d_{23}d_{31}, \quad (32)$$

where $\rho_1 \otimes \rho_1 = \rho_1(p_1)\rho_1(p_2)$ and the $d_{ij} = d(q_{ij})$ are functions of the 4-momentum differences q_{ij}.

While in principle calculable from a given density matrix, the functions d_{ij} are usually parametrized in a plausible and/or convenient way, such as Gaussian ($d_{ij} = \exp(-r^2 q_{ij}^2)$), exponential ($d_{ij} = \exp(-r q_{ij})$) and power-law parametrizations ($d_{ij} = q_{ij}^{-\alpha}$). Since the Gaussian does not give viable fits, only the latter two parametrizations are used below.

In Fig. 1, we show the results of a preliminary[4] comparison of the normalized third order cumulant ΔK_3 (29) obtained from UA1 minimum-bias data ($\bar{p}p$-reactions at $\sqrt{s} = 630$ GeV) with QS-predictions (32). Besides the GHP-sum topology used in (b), we show in (a) a separate analysis using the "GHP max" topology [6], which bins triplets according to the largest of the three momentum differences, $\max(q_{12}, q_{23}, q_{31})$. Fit parameter values used for the respective power law and exponential parametrizations were taken from the QS fit (31) to ΔK_2 obtained from the same data sample. All theoretical points shown are determined only up to an additive constant, so that the curves may be shifted up and down. It is clear, though, that the shape of third-order cumulant data measured differs appreciably from that predicted by the QS formulae and parameter values from ΔK_2. This conclusion holds independently of the topology used and of the functional form taken for d.

The results of this analysis may appear, at first sight, to contradict the conclusion [10], based on an earlier UA1 paper [12], that QS theory was compatible with higher-order *factorial moments*. The apparent discrepancy is explained by pointing out that the recent improvement of measurement techniques have permitted the present direct measurements of *cumulants*, which are considerably more sensitive than moments. The latter are dominated numerically by the combinatorial background of lower order correlations and thus contain mostly redundant information.

[4] This preliminary analysis is based on like-sign particles in a restricted phase space region with good detector acceptance. The analysis is currently being extended to an enlarged phase space region.

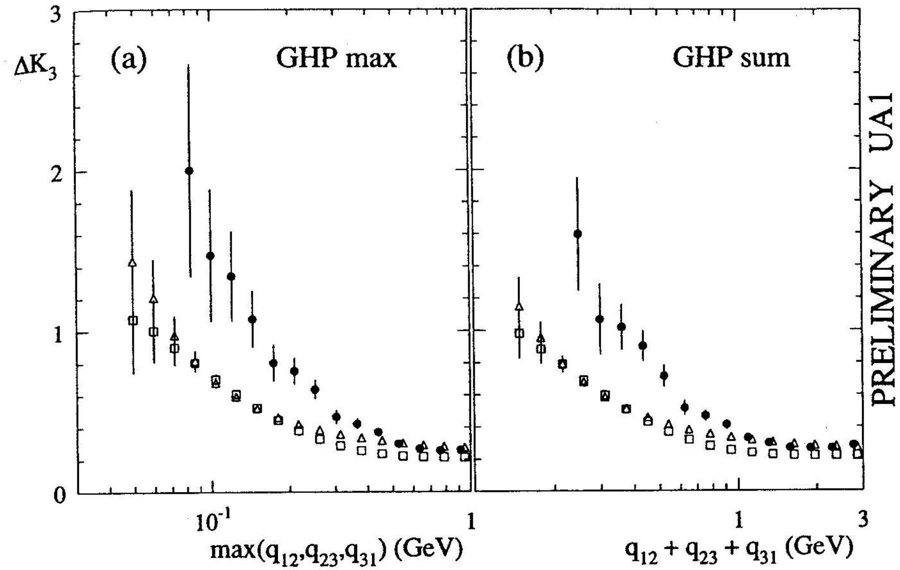

Fig. 1. Third-order differential cumulants $\Delta K_3(\epsilon)$ integrated with GHP max (a) and GHP sum (b) topologies [8]. Filled circles represent UA1 minimum-bias data, while open symbols show predictions for ΔK_3 from QS theory and parameter values obtained from fits to ΔK_2. Open triangles are predictions based on the QS power-law parametrization; open squares are QS exponential predictions.

5. Final remarks

The use of correlation integrals permits much accurate measurements and hence will likely reveal more detailed structure of the underlying dynamics. In particular, cumulants are promising and sensitive tools to obtain refined insight into various production mechanisms.

Greater accuracy requires, however, that possible biases be understood on a deeper level than before. One such bias arising generally in the measurement of correlations is due to the *finite* size of event samples and can have quite noticeable size. In practice, one has to use *unbiased* estimators, witch correct for this effect (cf. [6]).

Another caveat is the comparsion of experimental correlation data with theoretical predictions, when the latter are given in terms of differentially normalized (reduced) correlation functions such as (31) and (32). *Experimentally*, one can never measure fully differential ratios; rather, the numerator and denominator are averaged over some bin of finite size Ω (however small) before the ratio is taken. The discrepancy to fully differential ratios can be quite substantial. A procedure to overcome this problem amounts to a Monte Carlo integration of a *theoretical correlation function* sampled according to the *experimental uncorrelated one-particle distribution*; this and other details are explained in [8].

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