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ORDERINGS OF NONCOMMUTING QUANTUM PHASE OPERATORS: UNDESIRABLE AND DESIRABLE RESULTS[1]

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We use the analogy between the photon annihilation and creation operators and the Susskind–Glogower exponential phase operators to introduce the quasidistributions of the cosine and sine operators.

1. Introduction

Besides the measurement of the phase shift and the operational approach to the phase difference operators, we may distinguish the formalisms of the realistic phase from those of the ideal optical phase. The realistic phase can be named also the phase from phase space distributions [1–6]. The most familiar phase space distributions are P, W, and Q functions. They are derived via orderings of photon annihilation and creation operators and, in fact, they represent only joint quasidistributions of the quadratures. Exactly, this is inevitable for P and W quasidistributions related to the normal and symmetric orderings of the annihilation and creation operators, respectively, but the Q quasidistribution related to the antinormal ordering is encountered in the simultaneous measurement of the quadratures. Consulting [7], we learn that quasidistributions can be defined via two sorts of orderings of the quadrature operators. Unfortunately, the standard and antistandard orderings of these operators lead to quasidistributions, which can be not only negative, but also imaginary.

The formalisms of the ideal optical phase seem to be unified due to the success of the Pegg–Barnett formalism [8,9], which has superseded the previous Susskind–Glogower formalism [10], and owing to the recognition of the antinormal ordering of the Susskind–Glogower exponential phase operators [11–13]. In this paper we intend to illustrate the similarities and differences between the antinormal ordering of the annihilation and creation operators and that of the exponential phase operators.

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2. Analogy

The Susskind–Glogower cosine and sine operators can be defined as

$$\widehat{\cos}\,\varphi = \mathrm{Re}[\widehat{\exp}(i\varphi)], \quad \widehat{\sin}\,\varphi = \mathrm{Im}[\widehat{\exp}(i\varphi)], \tag{1}$$

where the exponential phase operator

$$\widehat{\exp}(i\varphi) = (\hat{a}\hat{a}^\dagger)^{-\frac{1}{2}}\hat{a}, \tag{2}$$

with $\hat{a}$ and $\hat{a}^\dagger$ the photon annihilation and creation operators. Our study is based on observations that the commutator

$$[\widehat{\exp}(i\varphi), \widehat{\exp}(-i\varphi)] = |0\rangle\langle 0| \tag{3}$$

is a projection operator similarly as $[\hat{a}, \hat{a}^\dagger] = \hat{1}$ and that the commutator

$$[\widehat{\cos}\,\varphi, \widehat{\sin}\,\varphi] = \frac{i}{2}|0\rangle\langle 0| \tag{4}$$

is similar to $[\mathrm{Re}\,\hat{a}, \mathrm{Im}\,\hat{a}] = \frac{i}{2}\hat{1}$.

In addition to the properties of the Susskind–Glogower phase operators which are exposed by the commutators, there are some which are related to associative algebra. Let us name

$$(\widehat{\cos}\,\varphi)^2 + (\widehat{\sin}\,\varphi)^2 = \hat{1} - \frac{1}{2}|0\rangle\langle 0|. \tag{5}$$

This relation is suspicious at first sight. In fact, in the measurement theory we can assign, to the Hermitian operator on the left- or right-hand side of (5), a random variable taking only the values of $\frac{1}{2}$ and 1. In a state of the physical system, where the photon number distribution is known to be $\{p(n)\}$, the sum of squared trigonometric operators takes the value of $\frac{1}{2}$ with the probability $p(0)$ and the value of 1 with the probability $1 - p(0)$. It is the possibility that the random variable assumes the value of $\frac{1}{2}$ that provokes. Although there is no need of ordering cosine and sine operators on the left-hand side of (5), we see that using (1), we sum the expressions

$$(\widehat{\cos}\,\varphi)^2 = \frac{1}{4}\left[\widehat{\exp}(i2\varphi) + \widehat{\exp}(i\varphi)\widehat{\exp}(-i\varphi) + \widehat{\exp}(-i\varphi)\widehat{\exp}(i\varphi) + \widehat{\exp}(-i2\varphi)\right], \tag{6}$$

$$(\widehat{\sin}\,\varphi)^2 = -\frac{1}{4}\left[\widehat{\exp}(i2\varphi) - \widehat{\exp}(i\varphi)\widehat{\exp}(-i\varphi) - \widehat{\exp}(-i\varphi)\widehat{\exp}(i\varphi) + \widehat{\exp}(-i2\varphi)\right], \tag{7}$$

which are ordered symmetrically in the exponential phase operators. But, if we order the operators $\widehat{\exp}(i\varphi)$, $\widehat{\exp}(-i\varphi)$ antinormally, we obtain the operators

$$[(\widehat{\cos}\,\varphi)^2]_{\mathcal{A}}^{CS} = \frac{1}{2}[\hat{1} + \widehat{\cos}(2\varphi)], \quad [(\widehat{\sin}\,\varphi)^2]_{\mathcal{A}}^{CS} = \frac{1}{2}[\hat{1} - \widehat{\cos}(2\varphi)]. \tag{8}$$

On summing, we get

$$\left[(\widehat{\cos}\,\varphi)^2 + (\widehat{\sin}\,\varphi)^2\right]_{\mathcal{A}}^{CS} = \hat{1}. \tag{9}$$

A similar use of the normal ordering leads to

$$\left[(\widehat{\cos}\,\varphi)^2 + (\widehat{\sin}\,\varphi)^2\right]_{\mathcal{N}}^{CS} = \hat{1} - |0\rangle\langle 0|. \tag{10}$$

In the measurement theory we assign to the Hermitian operator of (10) a random variable taking the values of 0 and 1. The possibility that the random variable assumes the value of 0 is undesirable. On the contrary, to the identity operator in (9), we assign the deterministic value of 1 [11].

The analogy between the operators $\hat{a}$, $\hat{a}^\dagger$, Re $\hat{a}$, Im $\hat{a}$ on the one hand and the operators $\widehat{\exp}(i\varphi)$, $\widehat{\exp}(-i\varphi)$, $\widehat{\cos}\,\varphi$, $\widehat{\sin}\,\varphi$ on the other hand is so far going that the terms the standard and antistandard orderings for the operators $\widehat{\cos}\,\varphi$, $\widehat{\sin}\,\varphi$ and the terms the normal and antinormal orderings for the operator $\widehat{\exp}(i\varphi)$, $\widehat{\exp}(-i\varphi)$ are natural. These orderings are given by the following rules of assignment:

Standard	$\sin^m\varphi\cos^n\varphi \to \widehat{\cos}^n\varphi\widehat{\sin}^m\varphi$
Antistandard	$\sin^m\varphi\cos^n\varphi \to \widehat{\sin}^m\varphi\widehat{\cos}^n\varphi$
Normal	$\exp(-im\varphi)\exp(in\varphi) \to \widehat{\exp}(-im\varphi)\widehat{\exp}(in\varphi)$
Antinormal	$\exp(-im\varphi)\exp(in\varphi) \to \widehat{\exp}(in\varphi)\widehat{\exp}(-im\varphi)$

In relation to the symmetric or Weyl ordering the four phase operators behave equally well. Nevertheless, just in this case the Pythagorean theorem, as applied to the unit circle, is not obeyed. In the only case of the antinormal ordering, results are desirable.

The eigenstates of the cosine operator are given by the expansion

$$|C\rangle = \begin{cases} \sqrt{\frac{2}{\pi}}\sqrt[4]{1-C^2}\sum_{n=0}^{\infty}U_n(C)|n\rangle & \text{for } C \in [-1,1], \\ 0 & \text{for } C \overline{\in} [-1,1], \end{cases} \tag{11}$$

where $U_n(C)$ are the Chebyshev polynomials of second kind [14]. In fact, no eigenstates can be the null vectors, but we add them having in mind the spectral density measure. The eigenstates of the sine operator are defined by the property

$$|S\rangle = i^{\hat{n}}|C = S\rangle, \tag{12}$$

where $\hat{n}$ is the number operator, $\hat{n} = \hat{a}^\dagger\hat{a}$. Let us note that the unitary operator $i^{\hat{n}} = \exp\left(i\frac{\pi}{2}\hat{n}\right)$ represents the rotation of the phase space by $\frac{\pi}{2}$.

Assuming

$$\varphi = \cos^{-1} C, \tag{13}$$

we remember that the eigenstates of the cosine operator are superpositions of the so-called phase states

$$|C\rangle = \frac{1}{i\sqrt[4]{1-C^2}}[\exp(i\varphi)|\varphi\rangle - \exp(-i\varphi)|-\varphi\rangle]. \tag{14}$$

The phase states $|\varphi\rangle$ have the property,

$$\langle\varphi|\varphi'\rangle = \frac{1}{4\pi} + \frac{1}{2}\delta(\varphi - \varphi') - i\frac{1}{4\pi}\mathcal{P}_\varphi\cot\left(\frac{\varphi - \varphi'}{2}\right), \tag{15}$$

where $\mathcal{P}_\varphi$ (principal value) denotes a generalized function of φ, which is the limit of the indicated function, but replaced by zero in a symmetric neighbourhood of $\varphi = \varphi'$ (mod 2π). From this we derive the following cumbersome formula

$$\langle C|S\rangle = \sqrt[4]{1-C^2}\sqrt[4]{1-S^2}\left[-\frac{1}{\pi}\mathcal{P}_C\frac{1}{C^2+S^2-1} + i\,\mathrm{sgn}(CS)\delta(C^2+S^2-1)\right], \quad (16)$$

where $\mathcal{P}_C$ relates to the zero points $C = \pm\sqrt{1-S^2}$. We introduce the quasidistributions of eigenvalues of the cosine and sine phase operators using the method of quantum characteristic function which was outlined in [15]. Quantum characteristic functions are defined by the scheme

$$C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \mathrm{Tr}\left\{\hat\rho\hat D^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right\}, \quad (17)$$

where $\hat\rho$ is the density (state) operator to be represented by the quasidistributions and

$$\hat D^{CS}_{\mathrm{st}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp(i\theta\,\widehat{\cos}\,\varphi)\exp(i\tau\,\widehat{\sin}\,\varphi), \quad (18)$$

$$\hat D^{CS}_{\mathrm{antist}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp(i\tau\,\widehat{\sin}\,\varphi)\exp(i\theta\,\widehat{\cos}\,\varphi), \quad (19)$$

$$\hat D^{CS}_{\mathcal{N}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp\left[\frac{-\tau+i\theta}{2}\widehat{\exp}(-i\varphi)\right]\exp\left[\frac{\tau+i\theta}{2}\widehat{\exp}(i\varphi)\right], \quad (20)$$

$$\hat D^{CS}_{\mathcal{A}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp\left[\frac{\tau+i\theta}{2}\widehat{\exp}(i\varphi)\right]\exp\left[\frac{-\tau+i\theta}{2}\widehat{\exp}(-i\varphi)\right], \quad (21)$$

$$\hat D^{CS}_{\mathcal{S}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp(i\theta\,\widehat{\cos}\,\varphi + i\tau\,\widehat{\sin}\,\varphi), \quad (22)$$

with subscripts st, antist, $\mathcal{N}$, $\mathcal{A}$, $\mathcal{S}$ denoting the standard, antistandard, normal, antinormal, and symmetric orderings, respectively. The respective quasidistributions can be obtained according to the scheme

$$\Phi^{CS}(C+iS) = \left[\mathcal{F}^{-1}\left(C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right)\right](C+iS), \quad (23)$$

where the Fourier transform

$$\left[\mathcal{F}^{-1}\left(C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right)\right](C+iS)$$
$$= \frac{1}{4\pi^2}\int\!\!\int \exp(-i\theta C - i\tau S)C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\,d\theta d\tau. \quad (24)$$

Quite generally we apply the prescription (23) in the domain of operators and we arrive at the operator densities

$$\hat\Phi^{CS}(C+iS) = \left[\mathcal{F}^{-1}\left(\hat D^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right)\right](C+iS), \quad (25)$$

namely

$$\hat\Phi^{CS}_{\mathrm{st}}(C+iS) = \langle C|S\rangle|C\rangle\langle S|, \quad (26)$$
$$\hat\Phi^{CS}_{\mathrm{antist}}(C+iS) = \langle S|C\rangle|S\rangle\langle C|, \quad (27)$$
$$\hat\Phi^{CS}_{\mathcal{A}}(\rho e^{i\varphi}) = 2\delta(\rho^2-1)|\varphi\rangle\langle\varphi|, \quad (28)$$

$$\hat\Phi^{CS}_{\mathcal{S}}(C+iS) = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)\sum_{m=0}^{n}(C+iS)^{n-m}R_m^{(1,n-m)}(C^2+S^2)|n\rangle\langle m|$$
$$+\frac{1}{\pi}\sum_{m=0}^{\infty}(m+1)\sum_{n=0}^{m-1}(C-iS)^{m-n}R_n^{(1,m-n)}(C^2+S^2)|n\rangle\langle m|$$
$$\text{for } C^2+S^2<1, \quad (29)$$
$$\hat\Phi^{CS}_{\mathcal{S}}(C+iS) = \hat 0 \text{ for } C^2+S^2>1, \quad (30)$$

where $R_m^{(\alpha,\beta)}(x)$ are the shifted Jacobi polynomials [14]. We have omitted the case of the normal ordering from among (26)–(30), because the operator density is an operator-valued generalized function here. Still generally, an application of (23) to (17) leads to the rule

$$\Phi^{CS}(C+iS) = \mathrm{Tr}\left\{\hat\rho\hat\Phi^{CS}(C+iS)\right\}. \quad (31)$$

The reconstruction of the original state operator can or cannot be accomplished according to a scheme

$$\hat\rho = \int\!\!\int \Phi^{CS}(C+iS)\hat\Delta^{CS}(C+iS)\,dCdS, \quad (32)$$

where

$$\hat\Delta^{CS}_{\mathrm{st}}(C+iS) = -\pi\frac{C^2+S^2-1}{\sqrt[4]{1-C^2}\sqrt[4]{1-S^2}}|S\rangle\langle C|, \quad (33)$$
$$\hat\Delta^{CS}_{\mathrm{antist}}(C+iS) = -\pi\frac{C^2+S^2-1}{\sqrt[4]{1-C^2}\sqrt[4]{1-S^2}}|C\rangle\langle S|, \quad (34)$$

$$\hat\Delta^{CS}_{\mathcal{N}}(C+iS) = |C+iS\rangle\langle C+iS|, \quad (35)$$

with $|C+iS\rangle$ a coherent phase state,

$$\hat\Delta^{CS}_{\mathcal{S}}(C+iS) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{nm}(C^2+S^2)\langle n|\hat\Phi^{CS}_{\mathcal{S}}(C+iS)|m\rangle|n\rangle\langle m|, \quad (36)$$

with

$$u_{nm}(C^2+S^2) = \pi\frac{(n+m+1)}{(n+1)(m+1)}(1-C^2-S^2). \quad (37)$$

In the case of the antinormal ordering the scheme (32) cannot be embodied, because the state of the physical system cannot be determined completely by the mere phase properties.

3. Unusual operator orderings

The analogy between the operators $\hat{a}$, $\hat{a}^\dagger$, Re $\hat{a}$, Im $\hat{a}$ and the operators $\widehat{\exp}(i\varphi)$, $\widehat{\exp}(-i\varphi)$, $\widehat{\cos}\ \varphi$, $\widehat{\sin}\ \varphi$ has led to the use of the terms the standard, antistandard, normal, antinormal, and symmetric orderings for the trigonometric and exponential phase operators. It has been found that the support of the quasidistributions related to the standard and antistandard orderings is the unit square, which is as ugly as that these quasidistributions can take imaginary values. In principle, the support of the quasidistributions related to the normal and symmetric orderings is the unit disc. As expected, the support for the antinormal ordering is the unit circle.

In the case of the normal ordering, the coherent phase state is represented by a Dirac delta function, a striking analogy to the normal ordering of the annihilation and creation operators. The analog of smoothing, which proceeds in the transition from this ordinary normal ordering to the ordinary antinormal ordering, cannot be obtained at some intermediate stages in the case of the exponential phase operators. The three stages available exhibit not only smoothing, but also some centrifugal trend. The joint quasidistribution of cosine and sine supported by the circumference is related to the antinormal ordering.

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