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OPTIMALIZATION OF PHASE SHIFT MEASUREMENT[1]

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Optimalization of phase shift measurement is addressed from viewpoint of quantum estimation theory. A probability operator measure (POM) describing the optimal measurement is derived. Afterwards we find input quantum states which minimize two different costs of error. These results are compared with several existing proposals of the phase shift measurement.

Papers about quantum phase measurement seems to be an evergreen of recent years. There are many suggestions of optimal measurements or optimal quantum states. In this overflow, however, the meaning of the word *optimal* is often vague. Here we draft the way from quantum description of an experimental setup over definition of optimality criteria to retrieval of the best measurement and the best appropriate quantum state.

The scheme considered here is a common Mach–Zehnder interferometer. Two modes of an input quantum state $|\psi_{in}\rangle$ are mixed at a lossless 50/50 beam splitter, then propagate along different paths and finally the second beam splitter produces an output state $|\psi(\theta_1,\theta_2)\rangle$. An unknown relative phase shift $\theta = \theta_1 - \theta_2$ is the quantity to determine. The transformation of the input state can be clearly expressed using generators of SU(2) algebra [1]. Let us define operators

$$\hat{J}_x = \frac{1}{2}(\hat{a}_1^+\hat{a}_2+\hat{a}_1\hat{a}_2^+),\ \hat{J}_y = \frac{1}{2i}(\hat{a}_1^+\hat{a}_2-\hat{a}_1\hat{a}_2^+),\ \hat{J}_z = \frac{1}{2}(\hat{a}_1^+\hat{a}_1-\hat{a}_2^+\hat{a}_2),\ \hat{N} = \hat{a}_1^+\hat{a}_1+\hat{a}_2^+\hat{a}_2,$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators of the input modes 1 and 2. Common eigenstates of $\hat{N}$ and $\hat{J}_z$ form a basis of two-mode Fock states $|jk\rangle_z = |j+k\rangle_1|j-k\rangle_2$, while common eigenstates of $\hat{N}$ and $\hat{J}_y$ form another basis, closely related to the Mach–Zehnder interferometer. The first beam splitter, the phase shift and the second beam splitter are represented in sequence with transformations

$$\hat{B}_1 = \exp(i\frac{\pi}{2}\hat{J}_x),\ \hat{P} = \exp(i\theta_2\hat{N} + i\theta\hat{N}/2 + i\theta\hat{J}_z),\ \hat{B}_2 = \exp(i\frac{\pi}{2}(\hat{N} - J_x)).$$

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Having neglected all factors independent of θ, the total transformation reads

$$|\psi(\theta)\rangle = \exp(i\theta\hat{N}/2)\exp(-i\theta\hat{J}_y)|\psi_{in}\rangle. \tag{1}$$

The first factor is often omitted [1,2] in order to get a simple tranformation of SU(2) type. However the transformation $\exp(-i\theta\hat{J}_y)$ yields the same output state as (1) only in following two cases: i) if we confine the set of all possible measurements to a particular detection method insensitive to the factor $\exp(i\theta\hat{N}/2)$, such as photon counting. Or ii) if we restrict the set of possible quantum states to density matrices factorised with respect to $\hat{N}$. A two-mode Fock state $|jk\rangle_z$ is a trivial example of such a factorised state. Generally the complete transformation (1) have to be used. Otherwise the omission of the first factor brings not simplification but complications: another spectrum of eigenvalues and 4π periodicity, which is intrinsic for particles with spin 1/2 but pointless for the photon interferometry. Further we discuss only a case of pure input states, $\hat{\rho}_{in} = |\psi_{in}\rangle\langle\psi_{in}|$. Usage of mixed states gives no improvement but complicates calculations too much .

We describe the estimation process with a *probability operator measure* (POM) [3]. Every non-negative additive Hermitian operator $\hat{\Pi}(\Delta)$ satisfying condition $\int_\Theta \hat{\Pi}(\theta)\mathrm{d}\theta = \hat{1}$ defines a measurement of a parameter θ with a probability distribution

$$P(\phi|\theta) = Tr\{\hat{\Pi}(\phi)\hat{\rho}(\theta)\}, \tag{2}$$

where ϕ is a value of the estimation of the unknown parameter θ. So as we can decide what kind of phase estimation is the best, let us define a *cost function* $C(\phi|\theta) = C(\phi-\theta)$. We use and compare two different functions: using delta function

$$C_\delta(\phi-\theta) = -\delta[(\phi-\theta) \bmod 2\pi] \tag{3}$$

means maximizing *peak likelihood* $P(\theta|\theta)$, while using quasiquadratic function

$$C_{sin}(\phi-\theta) = 4\sin^2\frac{\phi-\theta}{2} \tag{4}$$

means minimizing *dispersion*, a 2π–periodic analogy of familiar variance $(\Delta\phi)^2$. Since the delta cost function suffers some significant flaws [4], the sine cost function generally evaluate the precision of measurement better. Now we consider a POM optimal if it minimizes an average cost of errors

$$\bar{C} = \int\int C(\phi,\theta)P(\phi|\theta)z(\theta)\mathrm{d}\theta\mathrm{d}\phi, \tag{5}$$

where probability density $z(\theta)$ represents our *a priori* knowledge about estimated parameter θ (often $1/2\pi$ i.e. uniform).

The technique of construction of an optimal POM is described in [3]. We omit interesting but a little longer derivation and state only results here. If the set of input states is restricted to states with fixed number of photons,

$$|\psi_{in}\rangle = \sum_{k=-j}^{j} c_k|jk\rangle_y, \tag{6}$$

the optimal POM has a form of a *covariant measurement* [3,5]

$$\hat{\Pi}_{cov}(\phi) = e^{-i\phi\hat{J}_y}\hat{\Pi}_{cov}(0)e^{i\phi\hat{J}_y}, \tag{7}$$
$$\hat{\Pi}_{cov}(0) = \frac{1}{2\pi}\sum_{k,l=-j}^{j}\gamma_k\gamma_l^*|jk\rangle_{yy}\langle jl|.$$

The kernel $\hat{\Pi}_{cov}(0)$ is matched to the input state through $\gamma_k = \operatorname{Arg} c_k$. This POM generates shift-invariant phase distribution

$$P(\phi|\theta) = \frac{1}{2\pi}\left|\sum_{k=-j}^{j}|c_k|e^{-ik(\theta-\phi)}\right|^2. \tag{8}$$

The covariant POM depends little on a particular choice of a cost function. It is optimal for a rather wide set of "reasonable" cost functions including (3) and (4) [5].

A *canonical measurement* represents another POM often considered optimal [2]. In the $\hat{J}_y$ basis it has a structure similar to the covariant measurement with a kernel

$$\hat{\Pi}_{can}(0) = \frac{1}{2\pi}\sum_{k,l=-j}^{j}|jk\rangle_{yy}\langle jl| \tag{9}$$

and yields phase distribution

$$P(\phi|\theta) = \frac{1}{2\pi}\left|\sum_{k=-j}^{j}c_k e^{-ik(\theta-\phi)}\right|^2. \tag{10}$$

The difference is obvious: the canonical measurement is not matched to the input state and hence it measures not the induced phase shift but an overall phase of the output state with respect to the $\hat{J}_y$ basis. When the coefficients c_k are not real and positive (e.g. for a coherent state), the canonical POM does not yield a correct phase shift distribution. This comparison also shows why measurements with mixed states are generally worse than with pure states – if particular components of a mixed state are not phase matched, no POM can be matched to such input and resulting non-coherent superposition yields worse phase resolution.

Having founded the best POM, the second part of the optimization is retrieval of the quantum state which minimizes the average cost (5). Unlike the probability operator measure, choice of the best input state depends on a chosen cost function. Eq.(8) shows that the phase distribution depends only on the probabilities $P_k = |c_k|^2$. For the delta cost function (3), the best input has a form

$$|\psi_j^{hom}\rangle = (2j+1)^{-1/2}\sum_{k=-j}^{j}e^{i\chi_k}|jk\rangle_y, \tag{11}$$

$\|\psi_{in}\rangle$		$\|\psi_j^{hom}\rangle$	$\|\psi_j^{cos}\rangle$	$\|j0\rangle_z$	$\|jj\rangle_z$
$P(\theta\|\theta)$	$\propto$	N	N	N	$N^{1/2}$
$\Delta\phi$	$\propto$	$N^{-1/2}$	N^{-1}	$N^{-1/4}$	$N^{-1/2}$

Table 1: Dependance of precision of the optimal measurement on used energy N.

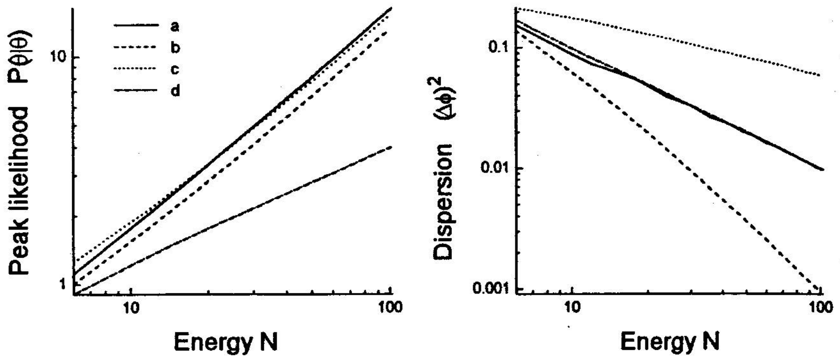

Figure 1: Dependance of peak likelihood and dispersion on used energy: (a) $|\psi_j^{hom}\rangle$, (b) $|\psi_j^{cos}\rangle$, (c) $|j0\rangle_z$, (d) $|jj\rangle_z$.

χ_k being arbitrary phases. For the sine cost function (4), the best input reads

$$|\psi_j^{cos}\rangle = (j+1)^{-1/2} \sum_{k=-j}^{j} e^{i\chi_k} \cos\frac{k\pi}{2(j+1)} |jk\rangle_y. \tag{12}$$

Attained precision for both states is compared in Tab.1 and Fig.1. There are also results for two other inputs: a Fock state with equal photon numbers in both the input ports $|j0\rangle_z = |j\rangle_1|j\rangle_2$, suggested recently in [2,6], and a Fock state $|jj\rangle_z = |2j\rangle_1|0\rangle_2$ with all photons in the input 1 and the input 2 empty. Structure of the states with respect to the $\hat{J}_y$ basis is displayed in Fig.2.

Now we leave off the constraint (6) and consider a general input state

$$|\psi_{in}\rangle = \sum_{2j=0}^{\infty} \sum_{k=-j}^{j} c_{jk} |jk\rangle_y, \tag{13}$$

index $2j$ means summing over both integer and half-integer values. Hereafter we have to use the full transformation (1). Having optimal POMs $\hat{\Pi}_j(\phi)$ for each of j-subspaces, it is natural to simply add them,

$$\hat{\Pi}_{sum}(\phi) = \sum_{2j=0}^{\infty} \hat{\Pi}_j(\phi). \tag{14}$$

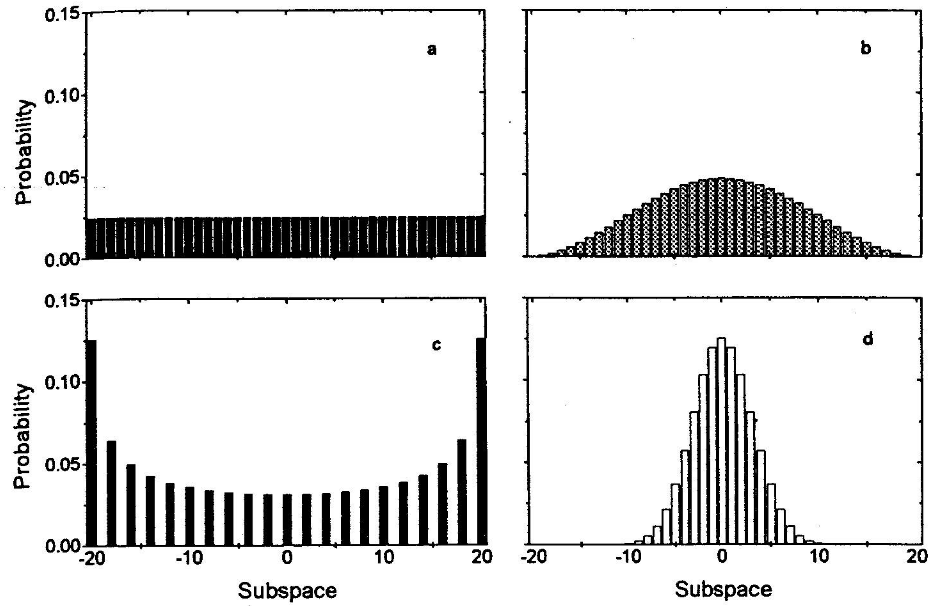

Figure 2: Structure of used input states: (a) $|\psi_j^{hom}\rangle$, (b) $|\psi_j^{cos}\rangle$, (c) $|j0\rangle_z$, (d) $|jj\rangle_z$.

However, in this manner contributions from component subspaces would be added only non-coherently,

$$P(\phi|\theta) = \frac{1}{2\pi} \sum_{2j=0}^{\infty} \left| \sum_{k=-j}^{j} |c_{jk}| e^{-ik(\theta-\phi)} \right|^2 . \tag{15}$$

The optimal POM can be constructed in the following way. Let operator $\hat{P}_k$ is a projector to the subspace appropriate for eigenvalue k of $\hat{N}/2 - \hat{J}_y$ operator and probability incident to this projection reads

$$P_k = \langle\psi_{in}|\hat{P}_k|\psi_{in}\rangle = \sum_{2j=k}^{\infty} |c_{j,j-k}|^2.$$

Then the optimality conditions are fulfiled by POM

$$\hat{\Pi}_{coh}(\phi) = e^{i\phi(\hat{N}/2-\hat{J}_y)} \hat{\Pi}_{coh}(0) e^{-i\phi(\hat{N}/2-\hat{J}_y)}, \tag{16}$$

$$\hat{\Pi}_{coh}(0) = \frac{1}{2\pi} \left\{ \hat{1} + \sum_{\substack{k,l=0 \\ k\neq l}}^{\infty} \frac{1}{\sqrt{P_k P_l}} \hat{P}_k |\psi_{in}\rangle\langle\psi_{in}| \hat{P}_l \right\}.$$

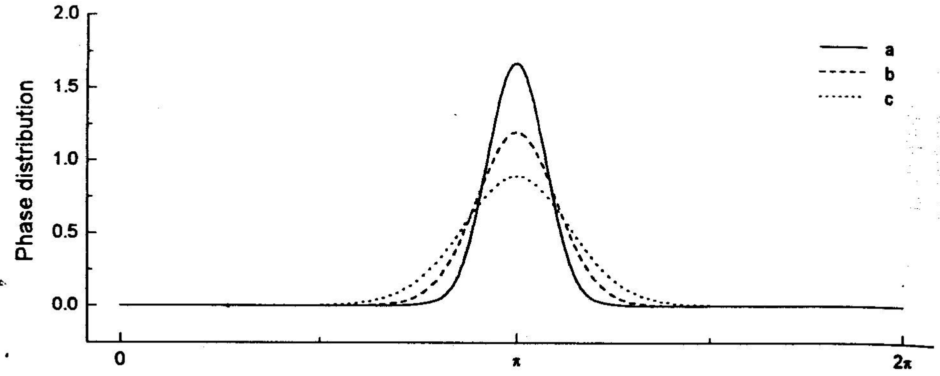

Figure 3: Phase distributions for differrent POMs applied to a coherent state $|\alpha\rangle_1|0\rangle_2, |\alpha|^2 = 10$: (a) $\hat{\Pi}_{coh}$, (b) $\hat{\Pi}_{sum}$, (c) photodetection.

Measurement described with this POM yields phase distribution

$$P(\phi|\theta) = \frac{1}{2\pi}\left|\sum_{k=0}^{\infty} P_k^{1/2} e^{ik(\theta-\phi)}\right|^2 \qquad (17)$$

with coherently compounded contributions from the component subspaces. Fig.3 shows the difference between these two POMs for a coherent input state. A considerably worse result for direct photodetection [7] is displayed for comparison too.

We have shown a formulation of the phase shift measurement in the framework of quantum estimation theory. Commonly used description of the Mach–Zehnder interferometer is in some respect inaccurate and sometimes may lead to confusion. The POM describing the optimal measurement is similar to the canonical measurement but have to be matched to a used input state. Mere canonical POM can give incorrect results in a general case. Finally if the setup is not restricted to a fixed photon number in the input, a POM better than a simple sum of measurements on component subspaces can be found. The states giving the best resolution for the optimal POM then appoint ultimate limits of the phase shift measurement.

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