acta physica slovaca vol. 46 No. 3, 451 – 456 June 1996

QUANTUM STATE ENGINEERING IN FINITE-DIMENSIONAL HILBERT SPACE[1]

A. Miranowicz†[2], W. Leoński†, S. Dyrting‡, R. Tanaś†

† *Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland*

‡ *Department of Physics, University of Queensland, Qld 4072 Australia*

Received 31 May 1996, accepted 7 June 1996

We give a recipe for how to generate various harmonic oscillator states formally defined in finite-dimensional Hilbert space.

1. Introduction

Recently, various harmonic-oscillator states defined in a finite-dimensional Hilbert space (FDHS) have aroused considerable interest (see, e.g., Refs. [1-4] and references therein). These studies are stimulated by a possible application of the new discrete Wigner formalisms [5] to quantum-state tomography of finite-dimensional systems and, on the other hand, by the popularity of finite-dimensional approaches to the phase problem (including the Pegg-Barnett phase formalism) [1,6].

Several representations and physical interpretations of the FDHS states have been suggested within atom optics and cavity quantum electrodynamics. Moreover, states constructed in the FDHS go over into the standard (i.e., infinite-dimensional) ones in the dimension limit. In this Communication, we propose a new generation scheme of the finite-dimensional harmonic oscillator states.

Our approach is a generalization of the one-photon state preparation method developed by Leoński and Tanaś [7,8]. We study models of a nonlinear interaction of a pumping classical mode with a cavity field in a nonlinear medium described as a higher-order nonlinear Kerr medium (multi-photon anharmonic oscillator).

Several schemes of state engineering have already been proposed (see Ref. [9] and references therein). These methods offer the possibility of generating arbitrary photon-number states. In particular, they can be used to prepare any state of the FDHS as a finite superposition of Fock states. However, as we know, a method of direct generation of the FDHS states has, as yet, not been found. Our scheme enables us to perform a direct preparation of a large class of harmonic-oscillator states formally defined in the FDHS.

[1] Presented at the **4th central-european workshop on quantum optics**, Budmerice, Slovakia, May 31 - June 3, 1996

[2] E-mail address: miran@phys.amu.edu.pl

2. Harmonic oscillator states in FDHS

We are interested in states constructed in the finite-dimensional Hilbert space of a harmonic oscillator. This space, denoted by $\mathcal{H}^{(s)}$, is spanned by $(s+1)$ Fock states which are complete, $\hat{1} = \sum_{n=0}^{s} |n\rangle\langle n|$, and orthogonal, $\langle n|m\rangle = \delta_{n,m}$. Equivalently, the space $\mathcal{H}^{(s)}$ is spanned by the phase states $|\theta_m\rangle$ which also form a complete and orthonormal basis. The phase states were applied in various finite-dimensional formalisms of the Hermitian optical phase operator (for review see Ref. [6]). On the other hand, the phase states can be used in the construction of a discrete Wigner function [5].

The generalized (finite-dimensional) annihilation operator is defined in $\mathcal{H}^{(s)}$ by $\hat{a}^{(s)} = \sum_{n=1}^{s} \sqrt{n}|n-1\rangle\langle n|$. As can readily be checked, the commutator $[\hat{a}^{(s)}, (\hat{a}^{(s)})^\dagger]$ differs from $\hat{1}$. So, the operators $\hat{a}^{(s)}$ and $(\hat{a}^{(s)})^\dagger$ are not related to the Weyl-Heisenberg algebra. Moreover, the Baker-Hausdorff identity does not hold for them. These properties of the finite-dimensional annihilation and creation operators really complicate the analytical approach to quantum mechanics in $\mathcal{H}^{(s)}$, including the explicit construction of various finite-dimensional states.

The coherent states $|\alpha\rangle_{(s)}$ in the $(s+1)$-dimensional Hilbert space of a harmonic oscillator can be defined in a Glauber sense by the action of the analogue of the Glauber displacement operator on the vacuum state,

$$|\alpha\rangle_{(s)} = \hat{D}^{(s)}(\alpha)|0\rangle, \quad \text{where} \quad \hat{D}^{(s)}(\alpha) = \exp\left\{\alpha(\hat{a}^{(s)})^\dagger - \alpha^*\hat{a}^{(s)}\right\}, \tag{1}$$

as was suggested by Bužek et al. in Ref. [1]. The displacement operator $\hat{D}^{(s)}(\alpha)$ is given in terms of the finite-dimensional annihilation and creation operators. The coherent states $|\alpha\rangle_{(s)}$ are close analogues of Glauber's (i.e., infinite-dimensional) coherent states $|\alpha\rangle$. They were introduced and discussed by Bužek et al. [1] and their number-state representation was found by Miranowicz et al. [3]. A numerical analysis of the photon-number statistics of the states (1) in comparison with the Poissonian photon statistics of the standard infinite-dimensional coherent states was given in Ref. [1]. A comparative study of the phase properties of the finite- and infinite-dimensional coherent states within the Pegg-Barnett phase formalism was presented in Ref. [3]. A thorough treatment of these states, together with their discrete number-phase Wigner functions was presented in Ref. [4]. The finite-dimensional coherent states (1) approach the standard infinite-dimensional coherent states for $|\alpha|^2 \ll s$ as was shown numerically in Ref. [3] and analytically in Ref. [4].

Let us define squeezed vacuum $|\zeta\rangle_{(s)}$ in the $(s+1)$-dimensional Hilbert space by analogy with the standard (i.e., infinite-dimensional) squeezed vacuum, namely by the action of the finite-dimensional squeeze operator on vacuum, i.e.,

$$|\zeta\rangle_{(s)} = \hat{S}^{(s)}(\zeta)\,|0\rangle, \quad \text{where} \quad \hat{S}^{(s)}(\zeta) = \exp\left\{\zeta(\hat{a}^{(s)})^{\dagger 2} - \zeta^*(\hat{a}^{(s)})^2\right\} \tag{2}$$

and $\zeta = |\zeta|\exp(i\varphi)$ is the complex squeeze parameter. Our finite-dimensional squeezed vacuum $|\zeta\rangle_{(s)}$ goes in the limit of $s \to \infty$ into the standard squeezed state $|\zeta\rangle$.

3. Scheme of state generation in FDHS

We consider a cavity with a nonlinear Kerr medium. The cavity field, which is initially in a vacuum state, is pumped by a train of short pulses (kicks) of the classical electromagnetic field at the frequency of the cavity field. The process is governed by the general time-dependent Hamiltonian $\hat{H}(t)$

$$\hat{H}(t) \quad = \quad \hat{H}_{\text{Kerr}} + \hat{H}_{\text{kicks}}(t) \tag{3}$$

in the form of an unperturbed system, $\hat{H}_{\text{Kerr}}$, and a small driving perturbation, $\hat{H}_{\text{kicks}}(t)$. The unperturbed (between the kicks) evolution of the cavity field, in the $(2N-1)$th-order nonlinear Kerr medium (or N-photon anharmonic oscillator) is modelled in the interaction picture by the Hamiltonian [10]:

$$\hat{H}_{\text{Kerr}} \quad = \quad \frac{\hbar\chi_N}{N!}\hat{a}^{\dagger N}\hat{a}^N \;\equiv\; \frac{\hbar\chi_N}{N!}\hat{n}(\hat{n}-1)\cdots(\hat{n}-N+1), \tag{4}$$

where $\hat{a}$ is the annihilation operator for the cavity field; $\hat{n} = \hat{a}^\dagger\hat{a}$ is the photon number operator; χ_N is proportional to the $(2N-1)$th-order nonlinear susceptibility of the medium, $\chi^{(2N-1)}$. The time-dependent Hamiltonian

$$\hat{H}_{\text{kicks}}(t) \quad = \quad \epsilon\hbar(\hat{a}^{\dagger M} + \hat{a}^M)f(t) \tag{5}$$

describes Mth-order parametric process driven by a sequence of short pulses of the classical field. The kick strength ϵ is small enough ($\epsilon < 1$) will be treated as the strength of perturbation. In general, $f(t)$ is an arbitrary real periodic function of t with the period T. We assume that the time T between the kicks is much longer than $2\pi/\omega$, where ω is the field frequency. Under this assumption, the short pulses of the pump field of the frequency ω can be modelled by delta functions, $f(t) = \sum_{n=0}^{\infty}\delta(t-nT)$. If $|\phi(0)\rangle$ is a state at $t=0$ then the state $|\phi(kT)\rangle$ after k kicks is given by

$$|\phi(kT)\rangle = \hat{U}^k|\phi(0)\rangle, \tag{6}$$

where the evolution operator $\hat{U}$ is generated by $\hat{H}(t)$ which evolves states from $t=0$ to $t=T$. For $\epsilon = 0$, the system (3) has a simple operator solution [10]. In order to find the eigenstates of $\hat{U}$ for $0 < \epsilon < 1$, we apply the generalized Rayleigh-Schrödinger perturbation theory, i.e. a generalization of time-independent perturbation theory for systems whose perturbations are in the form of periodic driving [11,12]. Our problem is equivalent to finding the Floquet states, or to diagonalizing the sum of the momentum-like operator and the Hamiltonian (3), $-i\hbar\frac{d}{dt} + \hat{H}(t)$, in the extended Hilbert space $\mathcal{H} \otimes L_2(0,T)$. We state that the degeneracy of the Kerr medium Hamiltonian (4) determines, under certain conditions, the dimension of the Hilbert space. In the next paragraphs, we will show that our systems modelled by (3), for properly chosen values of M and N, evolve into various states of the finite-dimensional Hilbert spaces.

3.1. Coherent state generation in FDHS

We apply the generalized Rayleigh-Schrödinger perturbation theory, described in Ref. [12], to the Hamiltonian (3), for $M = 1$ and $N = s + 1$. We find that the system evolves at $t = kT$ into the state

$$|\phi(kT)\rangle = \sum_{n=0}^{s} C_n^{(s)}|n\rangle + \epsilon C_{s+1}^{(s)}|s+1\rangle + \mathcal{O}(\epsilon^2), \tag{7}$$

where the superposition coefficients $C_n^{(s)} = \langle n|\phi(kT)\rangle$ for $n = 0, ..s$ are

$$C_n^{(s)} = \frac{s!}{s+1}\frac{(-1)^n}{\sqrt{n!}}\sum_{m=0}^{s}\exp\left(ikx_m\epsilon c_0\right)\frac{\mathrm{He}_n(x_m)}{[\mathrm{He}_s(x_m)]^2} \tag{8}$$

and for $n = s + 1$ is

$$C_{s+1}^{(s)} = \sqrt{s+1}BC_s^{(s)} = (-1)^s B\sqrt{\frac{s!}{s+1}}\sum_{m=0}^{s}\frac{\exp(ikx_m c_0\epsilon)}{\mathrm{He}_s(x_m)}. \tag{9}$$

Here, $x_m \equiv x_m^{(s+1)}$ are the roots of the Hermite polynomial of order $(s+1)$, $\mathrm{He}_{s+1}(x_m) = 0$. The coefficient B is defined by

$$B = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{c_n}{n+a}, \quad \text{where} \quad c_n = \int_0^T f(t)\exp\left(-i2\pi n\frac{t}{T}\right)\mathrm{d}t \tag{10}$$

is the Fourier transform and a is $\chi_{s+1}T/(2\pi)$. The Eqs. (10) are valid for arbitrary real periodic function $f(t)$ with the period T. By comparing the coefficients (9) with those in Fock expansion of the finite-dimensional coherent states (1) (see Ref. [3]) and omitting terms proportional to ϵ, we find that

$$|\alpha = -ikc_0\epsilon\rangle_{(s)} = |\phi(kT)\rangle + \mathcal{O}(\epsilon), \tag{11}$$

i.e., the state created in the process governed by the Hamiltonian (3) for $M = 1$ and $N = s + 1$ is the finite-dimensional coherent state.

3.2. Squeezed vacuum generation in FDHS

Finite-dimensional squeezed vacuum (2) can be generated in the process governed by the Hamiltonians (4) for $N = s + 1$ and (5) for $M = 2$. Alternatively, instead of Hamiltonian (4), one can choose $\widehat{H}_{\mathrm{Kerr}}$ with lower nonlinearity:

$$\widehat{H}_{\mathrm{Kerr}} = \frac{\hbar\chi_{\bar{s}+1}}{(\bar{s}+1)!}\widehat{n}(\widehat{n}-2)\cdots(\widehat{n}-2\bar{s}), \tag{12}$$

where $\bar{s} = [\![s/2]\!]$ in terms of the Entier function. For brevity, we present only the results for the driving function $f(t)$ in terms of delta functions. As in §3.1, we apply the time-dependent perturbation method briefly discussed in §2. We find

$$|\phi(kT)\rangle = \sum_{n=0}^{\bar{s}} C_{2n}^{(s)}|2n\rangle + \epsilon C_{2\bar{s}+2}^{(s)}|2\bar{s}+2\rangle + \mathcal{O}(\epsilon^2), \tag{13}$$

where $\zeta = -ik\epsilon$. The superposition coefficients $C_{2n}^{(s)} = \langle 2n|\phi(kT)\rangle$ for $n = 0, .., \bar{s}$ are

$$C_{2n}^{(s)} = (-1)^n\frac{(2\bar{s})!}{\sqrt{(2n)!}}\sum_{m=0}^{\bar{s}}\exp(i|\zeta|x_k)\frac{\mathrm{G}_n(x_m)}{\mathrm{G}_{\bar{s}}(x_m)\mathrm{G}'_{\bar{s}+1}(x_m)}, \tag{14}$$

and

$$C_{2(\bar{s}+1)}^{(s)} = 2^{-\bar{s}-1}\sqrt{(2\bar{s}+1)(2\bar{s}+2)}C_{2\bar{s}}^{(s)}. \tag{15}$$

The polynomials $\mathrm{G}_n(x)$ in Eq. (14) are given by the recurrence formula $\mathrm{G}_{n+1}(x) = \mathrm{G}_n(x) - 2n(2n-1)\mathrm{G}_{n-1}(x)$ together with $\mathrm{G}_0(x) = 1$ and $\mathrm{G}_1(x) = x$. Here, $x_k \equiv x_k^{(s+1)}$ are the roots of the polynomial $\mathrm{G}_{s+1}(x)$. The same superposition coefficients (14) appear in the Fock expansion of the state (2) if the terms proportional to ϵ are omitted. We conclude that

$$|\zeta = -ik\epsilon\rangle_{(s)} = |\phi(kT)\rangle + \mathcal{O}(\epsilon), \tag{16}$$

viz. the system, described by the effective Hamiltonian (3) for $M = 2$ and a given N evolves into the state (13) which is the N-dimensional squeezed vacuum.

3.3. Fock state generation

As was shown by Leoński and Tanaś in Ref. [7], the one-photon Fock state can be obtained in the special case of our model studied in §3.1, i.e., for the cavity filled with the 3rd-order nonlinear Kerr medium described by the Hamiltonian $\widehat{H}_{\mathrm{Kerr}} = \frac{1}{2}\chi_2\widehat{n}(\widehat{n}-1)$. Here, the general perturbation solution (7) reduces to the simple form of the two-dimensional coherent state

$$|\phi(kT)\rangle = \cos(k\epsilon)|0\rangle - i\sin(k\epsilon)|1\rangle + \mathcal{O}(\epsilon) \tag{17}$$

for $f(t)$ in the form of delta functions with $c_0 = 1$. It is seen that the state (17) can reach the single-photon Fock state if the amplitude of the kicks and the time between the kicks is appropriately chosen. Similarly, the two-photon Fock state can be reached in the special case of process studied by us in §3.2, namely, for the Hamiltonian $\widehat{H}_{\mathrm{Kerr}} = \frac{1}{3!}\chi_3\widehat{n}(\widehat{n}-1)(\widehat{n}-2)$ or simpler $\widehat{H}_{\mathrm{Kerr}} = \frac{1}{2}\chi_2\widehat{n}(\widehat{n}-2)$. In these cases, the solution (13) takes the form

$$|\phi(kT)\rangle = \cos(\sqrt{2}k\epsilon)|0\rangle - i\sin(\sqrt{2}k\epsilon)|2\rangle + \mathcal{O}(\epsilon), \tag{18}$$

which is the three-dimensional squeezed vacuum if we neglect all terms proportional to ϵ. An N-photon Fock state can be generated in a Kerr medium, described by Hamiltonian $\widehat{H}_{\mathrm{Kerr}}$ degenerated at $n = 0$ and N and combined with the Nth-order parametric process modelled by (5). In general, for $N, M > 2$, the evolution of the optical system leads to a higher-order squeezed state in finite-dimensional Hilbert space.

4. Conclusion

We have given a recipe for how to generate various harmonic oscillator states formally defined in finite-dimensional Hilbert space. We believe that our method is the first which offers the possibility of direct generation of a large class of finite-dimensional harmonic oscillator states. To set an example, we have applied our method to prepare finite-dimensional coherent states and squeezed vacuum, and Fock states. This scheme can readily be applied in the generation of other harmonic oscillator states of the finite-dimensional Hilbert space, including displaced number states, squeezed and higher-order squeezed states, Schrödinger male and female cats, kitten states, phase coherent states, etc.

The problem of generating arbitrary quantum states of the electromagnetic field plays an essential role in quantum optics. States which can be directly generated in some systems are particularly important. We believe that our generation scheme emphasizes the real physical significance of the states mathematically constructed in finite-dimensional Hilbert space.

Acknowledgements We thank G. J. Milburn, T. Opatrný, and K. Piątek for valuable discussions. This work was carried out within the KBN Projects No. 2 P03B 188 8 and No. 2 P03B 128 8. One of us (A.M.) gratefully acknowledges the Fellowship of the Foundation for Polish Science.

References

[1] V. Bužek, A.D. Wilson-Gordon, P.L. Knight, W.K. Lai: *Phys. Rev. A* **45** (1992) 8079;

[2] P. Figurny, A. Orłowski, K. Wódkiewicz: *Phys. Rev. A* **47** (1993) 5151; L.M. Kuang, X. Chen: *Phys. Rev. A* **50** (1994) 4228; L.M. Kuang, F.B. Wang, Y. G. Zhou: *J. Mod. Opt.* **41** (1994) 1307; G. Gangopadhyay: *J. Mod. Opt.* **41** (1994) 525;

[3] A. Miranowicz, K. Piątek, R. Tanaś: *Phys. Rev. A* **50** (1994) 3423;

[4] T. Opatrný, A. Miranowicz, J. Bajer: *J. Mod. Opt.* **43** (1996) 417;

[5] U. Leonhardt: *Phys. Rev. Lett.* **74** (1995) 4101; J.A. Vaccaro, D.T. Pegg: *Phys. Rev. A* **41** (1990) 5156; T. Opatrný et al.: *Phys. Rev. A* **52** (1995) 2419; W.K. Wootters: *Ann. Phys.* **176** (1987) 1; O. Cohendet et al.: *J. Phys. A* **21** (1988) 2875;

[6] R. Tanaś, A. Miranowicz, Ts. Gantsog: *Progress in Opt.* **35** (1996) 355;

[7] W. Leoński, R. Tanaś: *Phys. Rev. A* **49** (1994) R20;

[8] W. Leoński: *Acta Phys. Slov.* **45** (1995) 383; W. Leoński, S. Dyrting, R. Tanaś: to be published;

[9] K. Vogel, V. M. Akulin, W. P. Schleich: *Phys. Rev. Lett.* **71** (1993) 1816; B.M. Garraway et al.: *Phys. Rev. A* **49** (1994) 535; S.Ya. Kilin, D.B. Horoshko: *Phys. Rev. Lett.* **74** (1995) 5206; J. Janszky et al.: *Phys. Rev. A* **51** (1995) 4191;

[10] C.C. Gerry: *Phys. Lett. A* **124** (1987) 237; V. Bužek, I. Jex: *Acta Phys. Slov.* **39** (1989) 351; M. Paprzycka, R. Tanaś: *Quantum Opt.* **4** (1992) 331;

[11] B. Wielinga, G.J. Milburn: *Phys. Rev. A* **49** (1994) 5042; H. Sambe: *Phys. Rev. A* **7** (1973) 2203; N.L. Manakov et al.: *Phys. Rep.* **141** (1986) 319;

[12] S. Dyrting, G.J. Milburn: *Phys. Rev. A* **48** (1993) 969;