QUASI-PERIODIC AND PERIODIC FIELD EVOLUTION IN FINITE-DIMENSIONAL HILBERT SPACE[1]

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We discuss the "time-evolution" of coherent states defined in finite dimensional Hilbert space. Two definitions of these states are considered: that of Glauber-like coherent states, and truncated coherent states. We pay attention to the periodic features in the evolution of these states.

1. Introduction

The most commonly used states in quantum optics are Glauber coherent states [1]. Quite recently Pegg and Burnett proposed phase operator [2] defined for the finite-dimensional Hilbert space (FDHS). Since strong interest in the problem of FDHS has appeared many authors adapted the definition of Glauber coherent states to the finite dimensional cases. It is possible to define FDHS coherent states in various ways. In this paper we will concentrate on two definitions. The one is based on the treatment of the coherent states as a result of the action of the displacement operator on the vacuum state. Of course, both the vacuum state and displacement operator are defined similarly as in the Glauber definition [1], but they are spanned in the FDHS. This approach was applied and discussed in the papers of Bužek *et al.* [3] and Miranowicz *et al.* [4]. The alternative attempt is based on the Glauber definition too. Nevertheless, for this case the vacuum state and the displacement operator are defined in infinite dimensional space. Then the Fock base expansion of this state is truncated. This method was developed by Kuang *et al.* [5]. In this paper similarly as in [6] those FDHS coherent states will be referred to as finite-dimensional Glauber coherent states (FDGCS) and truncated coherent states (TCS), respectively. Moreover, we will show that the FDHS coherent states evolve in a various way depending on the way they were defined, concentrating on their quasi-periodic and periodic behaviors.

2. Definitions of the FDHS coherent states

As we have mentioned earlier, Kuang et al. [5] defined the coherent states in FDHS by the truncation of the Fock expansion of the Glauber infinite-dimensional coherent state $|\bar{\alpha}\rangle(\infty)$. This method is equivalent to the action of a non-unitary operator $\exp(\bar{\alpha}\hat{a}^\dagger)$ on the vacuum state. Of course, proper normalization should be applied. Those states are referred to in this paper as *truncated coherent states* (TCS). We can rewrite the expansion of this state as in [6]:

$$|\bar{\alpha}\rangle_{(s)} = \mathcal{N}^{(s)}\exp(\bar{\alpha}\hat{a}^\dagger)|0\rangle = \sum_{n=0}^{s}\exp(in\phi)b_n^{(s)}|n\rangle,$$
(1)

where

$$b_n^{(s)} = \mathcal{N}^{(s)}|\bar{\alpha}|^n (n!)^{-1/2}.$$
(2)

The normalization constant $\mathcal{N}$ can be written in the following form:

$$\mathcal{N}^{(s)} = \left(\sum_{n=0}^{s}\frac{|\bar{\alpha}|^{2n}}{n!}\right)^{-1/2}$$
$$= \{(-1)^s L_s^{-s-1}(|\bar{\alpha}|^2)\}^{-1/2}.$$
(3)

The quantity L_s^{-s-1} appearing in (3) is a generalized Laguerre polynomial. Obviously, the dimension of the Hilbert space is equal to $(s+1)$.

The other type of FDHS coherent states are those referred to as *finite-dimensional Glauber coherent states*. They are defined in the same way as the appropriate states in infinite-dimensional Hilbert space, but all of the states and the Glauber unitary displacement operator $\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ are defined in the FDHS. Thus, for the $(s+1)$-dimensional Hilbert space , the expansion of the FDGCS can be written in the form [5]:

$$|\alpha\rangle_{(s)} = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle =$$
$$= \sum_{n=0}^{s}\exp(in\phi)c_n^{(s)}|n\rangle,$$
(4)

where

$$c_n^{(s)} = \frac{s!}{s+1}(n!)^{-1/2}(-i)^n\sum_{k=0}^{s}\exp(iz_k|\alpha|)He_n(x_k)He_s^{-2}(x_k).$$
(5)

The factors $He_n(x_k)$ are modified Hermite polynomials and x_k are the roots of the Hermite polynomial of order $(s+1)$: $(He_{(s+1)}(x_k) = 0)$. To obtain $He_n(x)$ from the Hermite polynomials H_n we use the following relation:
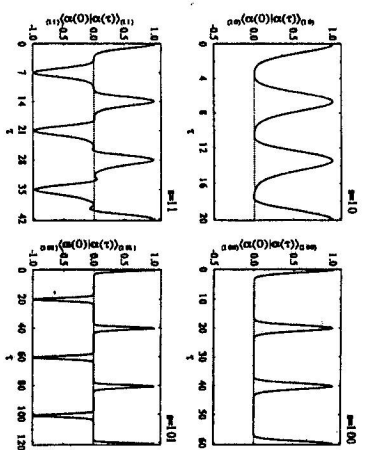
$$He_n(x) = 2^{-n/2}H_n(x/2^{1/2}).$$
(6)

Fig. 1. The scalar product $_{(s)}\langle\alpha(0)|\alpha(\tau)\rangle_{(s)}$ for $s = 10, 11, 100, 101$. Dotted line — TCS, solid line — FDGCS.

3. Quasi-periodicity and periodicity for the FDHS coherent states

In this paper we are interested in the quasi-periodic and periodic behaviors of the FDHS coherent states. At this point we should clarify what kind of periodicity we are interested in. Namely, we can treat the FDHS as the space in which the state evolves where the quantity α plays the role of generalized time t. In consequence, we can observe some periodicity for some states defined in FDHS as the function of the generalized time α. Hence we will use the two quantities α and the time t equivalently. From the definitions of the coefficients $b^{(s)}$ and $c^{(s)}$ (eqns.(2) and (5)) we see that periodic functions of α appear for the FDGCS coefficient $c^{(s)}$. The coefficients $b^{(s)}$ do not exhibit such periodic properties. Therefore, we will concentrate mainly on the FDGCS.

As it is seen from eqns.(4) and (5) FDGCS can be expressed as a sum of $\cos(x_k\alpha)f(x_k)$ and $\sin(x_k\alpha)g(x_k)$, where the factor x_k is a root of the Hermite polynomial He. Due to symmetry we include the positive x_k only, i.e. $0 < x_1 < x_2 < \ldots$. We are searching for the period $T = \alpha$ fulfilling the relation

$$|\alpha(t)\rangle_{(s)} \simeq |\alpha(t+T)\rangle_{(s)}$$
(7)

Due to the properties of the roots of the Hermite polynomials we can write (particularly for even s):

$$T \simeq \frac{2k\pi}{x_k}.$$
(8)

Using the appropriate approximation for the roots of the Hermite polynomials we obtain the following formula for the period T:

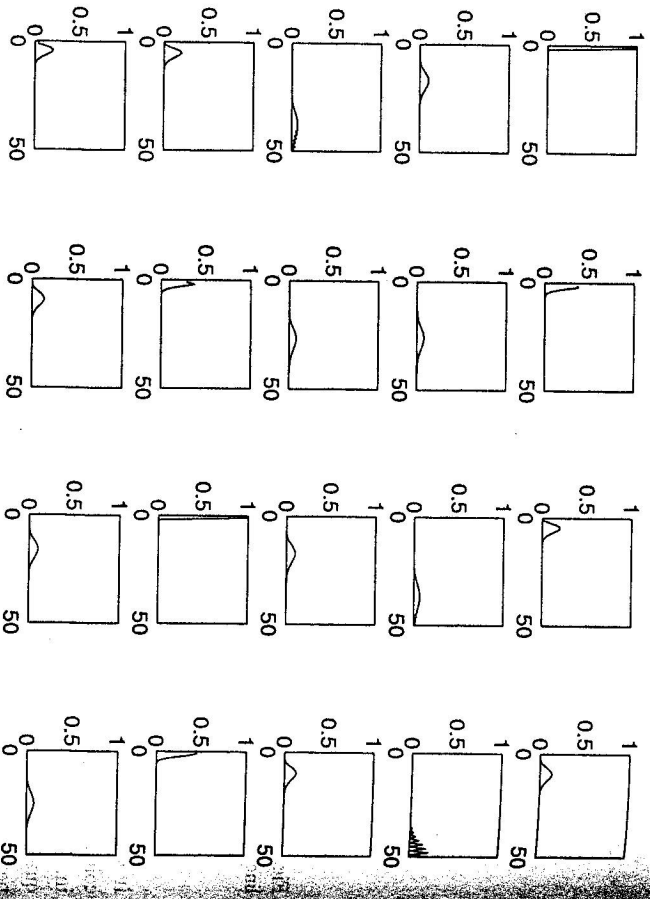$$T \simeq (1 + \delta_{s,odd})\sqrt{4s+6}.$$
(9)

Fig. 2. Number state distribution for various values of $\tau \equiv \alpha = 0, 1, \cdots, 20$ and $s = 50$.

As we will show, the value of T obtained from (9) is particularly close to the exact value of the period for even s.

To show the occurrence of the periodicity of the system (α is treated as a scaled time) we plot the product of two FDHS coherent states $(s)\langle \alpha(t_1)|\alpha(t_2)\rangle_{(s)}$. Assuming that $\alpha \in \Re$ this product becomes

$$(s)\langle \alpha(t_1)|\alpha(t_2)\rangle_{(s)} = f(t_2 - t_1) = f(\tau) \qquad (10)$$

Thus, Fig.1 shows this product as a function of the time τ. We see that for short time regime both FDGCS and TCS behave almost identically. Nevertheless, for longer times a significant difference between these two states occurs. TCS reaches its final value remaining constant, whereas FDGCS exhibits quasi-periodic behavior. Moreover, we see that for higher values of the space dimension ($s + 1$) this quasi-periodicity becomes almost ideally periodic. Fig.1 shows the "time-evolution" of the product (10) for both even and odd values of s. It is seen that for even s the periodicity is weaker - after each quasi-period additional, nonperiodic features become more and more visible.
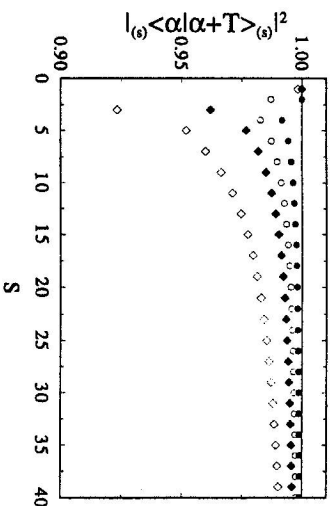
Nevertheless, they are negligible for higher values of s. In addition, for odd s the value of the quasi-period T is approximately two times greater than for its even counterpart. This fact originates from the "phase reversal effect". For this case the value of the scalar product reaches -1. We see that the product reaches the same value as for the initial time but with the opposite sign, contrary to the even s where the product reaches approximately the same value and sign as for the initial time. Moreover, for some values of τ the scalar product is equal to zero, so the initial state and the state for the time τ are orthogonal.

To show the behavior of the FDGCS during the "time-evolution" we plot (Fig.2) the number states distribution of the state for various values of α (treated as the time τ). We start from the vacuum state $|0\rangle$ and increase the value of α. We observe the wave-packet style distribution moving towards higher values of n. As this "packet" reaches the border of our space interference effects occur. This is an effect of the reflection of the "packet" from the border of the space. After this reflection the packet moves towards $n = 0$. During this motion it changes its shape and becomes a vacuum state distribution as for $\tau = 0$. This effect can be referred to as "ping effect". For further times we observe the same behavior as for $\tau < T$ (for $s = 50$ the periodicity is nearly ideal).

Fig.3 shows the function $|_{(s)}\langle \alpha(t)|\alpha(t + T)\rangle_{(s)}|^2$ plotted for various values of s. We have included plots corresponding to various methods of finding the period T. We compare the results for T found numerically and those derived from our approximate formula (9). We see that for $s = 1, 2$ the FDGCS exhibits ideal periodic behavior. Nevertheless, as the dimension of the space increases this periodicity changes to quasi-periodicity. This is a result of the form of the Fock-expansion of the FDGCS. The coefficients $c_n^{(s)}$ are defined by the sum of the periodic functions. The "frequencies" inside these functions are determined by the roots of the Hermite polynomials. For the cases of $s > 2$ the "frequencies" are not identical and their quotient is not a rational number. In consequence periodicity does not occur. Nevertheless, as s increases the

$$|_{(s)}\langle \alpha|\alpha + T\rangle_{(s)}|^2$$

Fig. 3. The quantity $|_{(s)}\langle \alpha|\alpha(\tau + T)\rangle_{(s)}|^2$ for various values of s. Circles correspond to even values of s, square marks — to odd values of s, empty marks — to approximate values of T, and filled marks — to T found numerically.

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quasi-periodic behavior tends to be ideally periodic. Moreover, Fig.3 shows that for even s we achieve better agreement between the approximate solution (9) and the T obtained from the numerical calculations. This fact agrees with our earlier discussion during derivation of the approximate formula for T.

4. Conclusions

We have discussed the *finite dimensional Glauber coherent states* (FDGCS) and the *truncated coherent states* (TCS). We have paid attention to the quasi-periodic and periodic properties of those states. We have shown that only the FDGCS exhibits those properties. Moreover, the behavior of this kind of states depends on the dimension of the space. For $s = 1, 2$ the states are periodic, whereas for higher values of s they are quasi-periodic. Nevertheless, as s increases their behavior tends to periodic. This tendency is more prominent for even values of s. Additionally, we have shown that for odd s *phase reversal effect* occurs. Moreover, we have shown that the dynamics of the FDGCS expressed in Fock-state basis exhibits the behavior that we have referred to as the *ping effect*. In this paper we derived an approximate analytical formula for the period T of the quasi-periodical behavior of the system. Our formula is particularly fulfilled for even s and agreement between our analytical result and the numerical result increases as the value of s becomes greater and greater.

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