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# ENTROPY OF PHASE MEASUREMENT[1]

**Z. Hradil**[2]
*Department of Optics, Palacký University 17. listopadu 50*
*772 07 Olomouc, Czech Republic*

**R. Myška**[3]
*Joint Laboratory of Optics, Palacký University and Czech Academy of Sciences,*
*17. listopadu 50, 772 07 Olomouc, Czech Republic*

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The content of phase information of an arbitrary phase–sensitive measurement is evaluated using the maximum likelihood estimation. The phase distribution may be characterized by the Kullback–Leibler relative information – a nonlinear functional of input quantum state. As an explicit example the measurement of quadrature operator is interpreted as quantum phase detection. The phase distribution achieves the ultimate resolution predicted by the Fisher information if exists. The method links several recently developed phase concepts such as Shapiro–Wagner phase measurement, marginal distribution of Q–function, "phase without phase" concept, adaptive phase or detection of phase shift in interferometry into a common framework of quantum estimation theory.

The problem of phase estimation on the basis of an arbitrary multiple measurement of phase–sensitive observable [1],[2],[3] may be sumarized in the following way. Assume for concreteness the quantum measurement of a quantum variable $\hat{Y}$ yielding discrete spectrum $|y_k\rangle$ enumerated for brevity by an index $k$. The purpose of phase detection is to determine a non–random c–number displacement parameter $\theta$ entering the unitary phase displacement transformation of quantum state as $|\psi(\theta)\rangle = e^{-i\theta\hat{N}}|\psi\rangle$, $\hat{N}$ being a Hermitian operator [4]. The variable $\theta$ represents the *true value* of the phase shift. The probability of finding $y_k$ by performing the measurement in transformed quantum state $|\psi(\theta)\rangle$ is given by quantum mechanics as $p(y_k,\theta) = |\langle\psi|e^{i\theta\hat{N}}|y_k\rangle|^2$. Knowing all these probabilities in dependence on the induced phase shift, an unknown phase shift may be inferred on the basis of multiple output data $y_1, y_2, \ldots, y_n$. Using the maximum likelihood estimation the estimator approximating the true phase is given as the phase

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[2]E-mail address: hradil@risc.upol.cz
[3]E-mail address: myskar@risc.upol.cz

 

maximizing the likelihood function $L(\phi) = p(y_1,\phi)p(y_2,\phi)\ldots p(y_n,\phi)$. The common envelope of all the phase histograms obtained by repeating the multiple measurement may be expressed as the conditional phase distribution of inferred phase shift $\phi$ when $\theta$ is true

$$P_{ML}(\phi|\theta) \propto \left\{ \prod_k [p_k(\phi)]^{p_k(\theta)} \right\}^n ; \tag{1}$$

$p_k(\theta) \equiv p(y_k,\theta)$ and index $k$ exhausts all the possible values appearing with nonzero probability. The number of samples $n$ is assumed to be sufficiently large in order to get statistically significant sampling. The distribution may be expressed using the relative information (Kullback–Leibler divergence) [5] $K(\phi|\theta) = \sum_k p_k(\theta)\ln p_k(\theta)/p_k(\phi)$ and Shannon entropy $H(\theta) = -\sum_k p_k(\theta)\ln p_k(\theta)$ as

$$P_{ML}(\phi|\theta) \propto e^{-n[H(\theta)+K(\phi|\theta)]}.$$

The estimation may be sometimes well approximated by the Gaussian distribution with the variance predicted by the Fisher information $I(\theta) = \sum_k [p'_k(\theta)]^2/p_k(\theta)$ as $\Delta\phi = 1/\sqrt{nI(\theta)}$, the prime denotes the derivative $p'_k(\theta) = dp_k(\phi)/d\phi|_{\phi=\theta}$. The general treatment will be demonstrated on the phase concepts already used in quantum theory.

### A. Quadrature measurement

Assuming the phase sensitive measurement of rotated quadrature operator $\hat{X}(\theta) = \frac{1}{\sqrt{2}}[\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}]$, the rotated quadrature eigenstates (variable $x$) appears with the probability $p(x,\theta) = |\langle\psi|x\rangle_\theta|^2$ depending on the actual phase of local oscillator $\theta$. Measuring the coherent state $|\alpha\rangle$, $\alpha = |\alpha|e^{i\varphi}$, the phase distribution inferred after n trials then reads [2]

$$P^X_{ML}(\phi|\theta') \propto \exp\{-2n|\alpha|^2[\cos\phi - \cos\theta']^2\}, \tag{2}$$

$\theta' = \theta - \varphi$. Measuring similarly the $\hat{P}(\theta)$ operator by an ordinary balanced homodyne detection scheme, the inferred distribution should be shifted by $\pi/2$ yielding

$$P^P_{ML}(\phi|\theta') \propto \exp\{-2n|\alpha|^2[\sin\phi - \sin\theta']^2\}. \tag{3}$$

### B. Adaptive measurement

The best phase resolution in the homodyne detection scheme (3) is achieved if the phase of local oscillator compensates the phase of signal field $\theta' = 0$. This may be done in controlled way by changing the phase of local oscillator until the registered value of $p$ is zero (in average). In this case the phasis are equal within the error predicted by the distribution

$$P^P_{ML}(\phi|\theta') \propto \exp\{-2n|\alpha|^2\sin^2\phi\}. \tag{4}$$

This idea was suggested by Wiseman [6] for phase shift measurement as an adaptive scheme, and also coincides with the prediction of so called "phase (measurement) without phase (states)" concept of Vogel and Schleich [7], motivated by geometrical considerations in phase space.



### C. Shapiro–Wagner phase

The phase sensitivity of phase distribution inferred on the basis either $\hat{X}$ or $\hat{P}$ operators are strongly shift–dependent. Particularly, the phase point $\theta' = 0$ of the best resolution of phase distribution (3) is the worst case for the phase distribution (2). This unpleasant property may be removed assuming detection of both the operators *simultaneously*. The phase distribution inferred after each single detection of $\cos\phi$ and $\sin\phi$ tends to the well-known Shapiro–Wagner phase concept [8],[9] (marginal distribution of Q–function)

$$P_{SW}(\phi) = \frac{1}{\pi}\int_0^\infty r dr \exp[-|r - |\alpha|e^{i(\phi-\theta')}|^2]. \tag{5}$$

On the other hand accumulation of such single detections followed by ML estimation yields the von Mises normal distribution on the circle [5]; $\kappa = 4n|\alpha|^2$

$$P_{ML}(\phi|\theta') = \frac{1}{2\pi I_0(\kappa)}\exp[\kappa\cos(\phi - \theta')]. \tag{6}$$

### D. Noh–Fougeres–Mandel scheme

In all the above mentioned examples the phase of signal field was determined against the phase of classical field of local oscillator. More general formulation anticipating the quantum nature of all the fields is given in the treatment suggested by Noh, Fougères and Mandel [10],[11]. In their Scheme 1 two fields are mixed on the beam splitter. Assuming the input coherent fields $\alpha_1$ and $\alpha_2$, the ML estimation tends to the inferred phase distribution as

$$P_{NFM}(\phi|\theta) \propto \prod_{j=3}^{6}[\bar{N}_j(\phi)]^{n\bar{N}_j(\theta)}, \tag{7}$$

where the average number of the particles on the output depends on phase difference $\varphi$ as

$$\bar{N}_{3,4}(\varphi) = \frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2\mp 2|\alpha_1||\alpha_2|\sin\varphi), \quad \bar{N}_{5,6}(\varphi) = \frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2\pm 2|\alpha_1||\alpha_2|\cos\varphi).$$

In all the above mentioned examples the phase resolutions may be compared in dependance on the total energy expenses $N$. Different strategies of phase estimation for $\theta' = 0$ are compared in Fig. 1. Optimized regime of distribution (3) is given by the distribution (4). Since $N = n|\alpha|^2$, the width defined as root-mean square for $N \to \infty$ is given as $\Delta\phi \propto 1/(2\sqrt{N})$. This estimation coincides with the ideal phase concepts [9] (curve $a$). On the other hand the phase distribution inferred using both the measurement of sin and cos functions gives the $\sqrt{2}$ times worse resolution. The phase distribution is denoted as curve ($b$) and is described by relations (5) or (6);

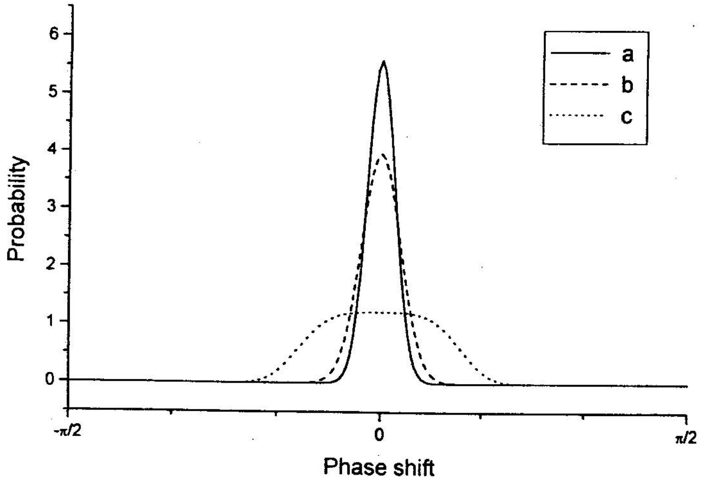


Fig. 1. Comparison of various strategies for phase estimation using quadrature measurement

$N = 2n|\alpha|^2$. The quadrature measurement of non-optimal component (2) yields the worst phase resolution with zero Fisher information (curve $c$.) In the last application of information theory, the detection and estimation will be distinguished carefully on the example of Mach–Zehnder interferometer beating the classical limit of phase resolution $1/\sqrt{N}$.

### E. Measurement beyond the classical limit

The phase resolution on the Mach–Zehnder interferometer may be reduced up to $1/N$ driving the interferometer with two Fock states containing equal numbers of photons [12]. Denoting $\theta$ the true value of induced phase shift the detected photocount distribution can be written as

$$P(2q|\theta) = \frac{(r-q)!}{(r+q)!}[P_r^q(\cos\theta)]^2,$$

where $P_r^q$ denotes the associated Legendre polynomials, $r$ being the number of photons on each input port, $2q$ being the difference of photons counted on each output port. The analysis of inferred phase shift will be given for special case of true phase $\theta = 0$, when the sharpest phase estimation is expected. In this case the value $q = 0$ only is registered. Nevertheless, this detection canot be interpreted as the inferrence of the zero phase shift. Particularly, after single detection, phase shift $\phi$ is inferred with the distribution $P_1(\phi|\theta = 0) \propto [J_0(r\phi)]^2$, since for $r$ large Legendre polynomials may be well approximated by Bessel functions $J_q$. Since the Bessel functions are not square integrabile, the estimation of phase resolution cannot be simply based to the "width" of respective distribution. More accurate analysis taking into account the nonintegrability



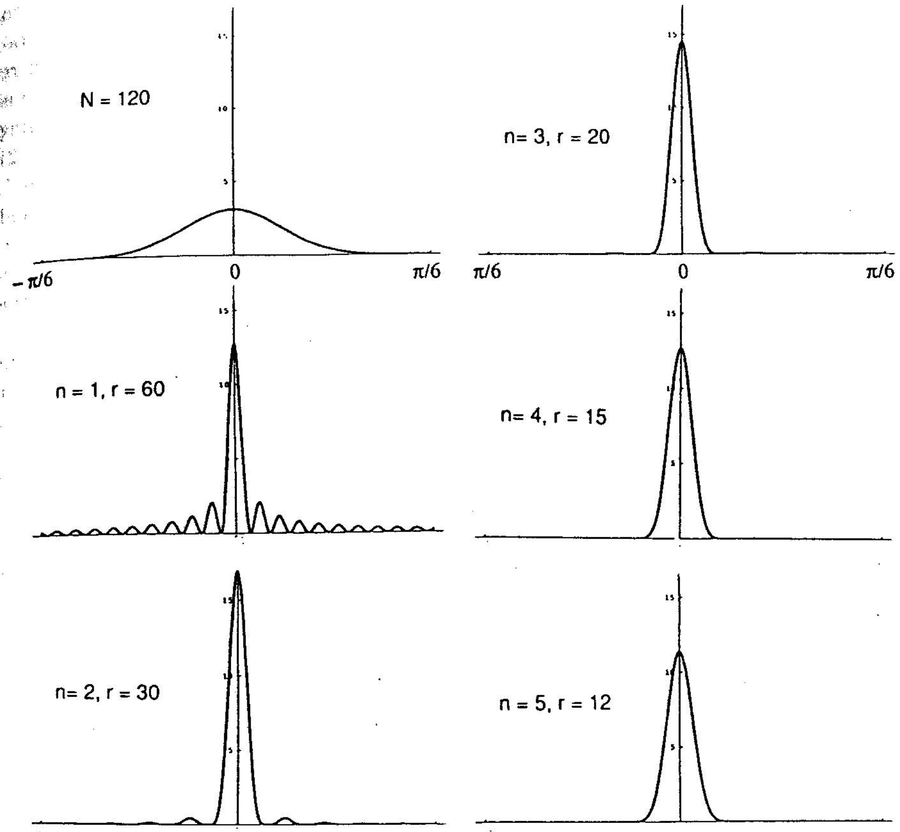


Fig. 2. Probability vs. inferred phase for Mach–Zehnder interferometer

of Bessel functions is necessary. Particularly, phase information after single counting of $q$ variable is worse than the classical measurement since

$$(\Delta\phi)^2 \approx (\frac{1}{r})^2 \frac{\int_0^{r\pi/2} dx J_0^2(x)x^2}{\int_0^{r\pi/2} dx J_0^2(x)} \propto 1/\ln r.$$

Nevertheless considerable improvement may be achieved in the multiple measurement scheme when the phase information is repeatedly updated during $n$ independent measurements $P_n(\phi|\theta = 0) \propto [J_0(r\phi)]^{2n}$. Taking into account square-nonintegrability of Bessel functions, the ultimate resolution predicted by Fisher information is $(\Delta\phi)^2 \propto 1/(nr^2)$ for $n \geq 4$, total number of particles being $N = 2rn$. Consequently, this result may be interpreted as proposed by Holland and Burnett [12] $(\Delta\phi)^2 \propto 4n/N^2$, but the optimum is achieved just for $n = 4$. If the strategy of phase detection is based mainly on the multiple repeating of single counting (i.e. $n$ is large) then the "classical" interpretation $\Delta\phi^2 \propto 2/(rN)$ seems to be more appropriate. This is demonstrated in Fig. 2, where the distribution $P_n(\phi|\theta = 0); n = 1, 2, 3, 4, 5$ and total number of particles $N = 120$ is compared to the classical scheme with one closed port and $N$ photons in the remaining port yielding the distribution of inferred phase shift $[(2N)!!/2\pi(2N-1)!!] \cos^{2N}(\phi/2)$ [1]. This explicit example demonstrates the difference between direct "measurement", when the desired information is derived immediately after the registration of measured variable, and quantum estimation, which optimizes also the procedure of data handling. This seemingly slight differences may apper as crucial, since the appropriate estimation may increase the resolution considerably. This point is sometimes overlooked in the literature [13] when the detection procedure only is optimized.

Interpretationally, the estimation theory completes the quantum measurement in the following way. The statistics of *detected* variable is determined immediately after the (single) registration. On the other hand statistics of *estimated* variable is found in two steps. At first the detected data are accumulated and then the desired information is derived by using an appropriate mathematical data treatment. Hence the statistics of measured variable is a linear functional of density matrix, whereas the statistics of inferred variable is given by a nonlinear one. Information content of the given phase sensitive measurement represents therefore an alternative to the semiclassical principle of correspondence for investigation of realistic phase concepts.

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## References

[1] Z. Hradil: *Phys. Rev. A* **51** (1995) 1870;

[2] Z. Hradil, R. Myška, T. Opatrný, J. Bajer: *Phys. Rev. A* **53** (1996) 3738;

[3] Z. Hradil, R. Myška, J. Peřina, M. Zawisky, Y. Hasegawa, H. Rauch: *Phys. Rev. Lett.* **76** (1996) 4295;

[4] C.W. Helstrom: *Quantum Detection and Estimation Theory* (Academic Press 1974, New York) chap. 8;

[5] R.C. Rao: *Linear Statistical Inference and Its Applications* (New York, Wiley, 1973) ;

[6] H.M. Wiseman: *Phys. Rev. Lett.* **75** (1995) 4587;

[7] W. Vogel, W. Schleich: *Phys. Rev. A* **44** (1991) 7642;

[8] J.H. Shapiro, S.S. Wagner: *IEEE J. Quantum Electron.* **20** (1984) 803;

[9] Z. Hradil: *Quantum Opt.* **4** (1992) 93;

[10] J.W. Noh, A. Fougères, L. Mandel: *Phys. Rev. Lett.* **67** (1991) 1426;

[11] J.W. Noh, A. Fougères, L. Mandel: *Phys. Rev. A* **45** (1992) 424;

[12] M.J. Holland, K. Burnett: *Phys. Rev. Lett.* **71** (1993) 1355;

[13] B.C. Sanders, G.J. Milburn: *Phys. Rev. Lett.* **75** (1995) 2944;