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DIRECT AND INDIRECT MEASURES OF PHASE[1]

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Recently, Torgerson and Mandel [1] have reported a disagreement between two schemes for measuring the phase difference of a pair of optical fields. We analyze these schemes and derive their associated phase-difference probability distributions, including both their strong and weak field limits.

1. Introduction

Lack of a canonical pair for the number and phase operators, $\hat{n}$ and $\hat{\phi}$, has led to much debate in quantum mechanics for many years now [2]. A more recent approach to the phase question involves concentrating more on what the experimentalist actually does: If his goal is a precision measurement then he can perform "any" measurement followed by data analysis to extract a classical parameter [3], for instance phase shift in an interferometer. Alternatively, he may mentally lump together the measurement and data analysis to construct an *operational phase operator* for his specific setup. This latter approach has been championed by Mandel and his coworkers [4]. The explicit construction and investigation of the properties of such phase observables can give us insight into the nature of quantum states.

Recently, Torgerson and Mandel [1] have compared two schemes for measuring the phase-difference between a pair of optical fields. They found that a direct scheme, where a signal is beat against a second one, and an indirect scheme, where the two signals are beat against a common local oscillator, yield *different* probability distributions for the measured phase-difference. In particular, they found that the schemes gave radically different distributions for very weak signals. Torgerson and Mandel take the conflicting results as evidence of the non-uniqueness of quantum phase. In their analyses ambiguous data is discarded. This post-selection procedure has generated some discussion in the literature [5]. We analyze these two schemes in the absence and presence of this post-selection and discuss its interpretation. We study limits for both strong and weak signals in these schemes and give closed form expressions for the phase-difference probability distributions.

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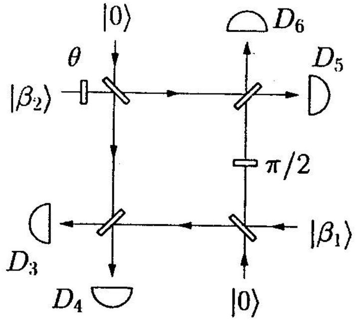

Fig. 1. Direct measurement scheme: A single eight-port homodyne detector is used to measure the phase difference between two optical fields β_1 and β_2 which are beat against each other directly. The input field β_2 is phase shifted by an angle θ. For each value of θ, the detectors D_3, D_4, D_5 and D_6 measure the photon count differences $n_{43} = n_4 - n_3$ and $n_{65} = n_6 - n_5$.

2. Schemes

In this section we briefly describe the direct and indirect schemes proposed in Ref. [1] and derive analytical expressions for the corresponding phase distributions

A. Direct measurement

The first scheme summarized in Fig. 1 uses a single eight-port homodyne detector to beat two coherent states $|\beta_1\rangle$ and $|\beta_2\rangle$ against each other. In this case the 'input' state is $|\psi_{\rm in}\rangle = |\beta_1\rangle \otimes |0\rangle \otimes |\beta_2\rangle \otimes |0\rangle$, and the 'output' state just before photodetection reads

$$|\psi_{\rm out}\rangle = \quad |\tfrac{1}{2}(\beta_1 - \beta_2 e^{i\theta})\rangle_3 \otimes |\tfrac{1}{2}(\beta_1 + \beta_2 e^{i\theta})\rangle_4 \otimes \\ |\tfrac{1}{2}(-i\beta_1 + \beta_2 e^{i\theta})\rangle_5 \otimes |\tfrac{1}{2}(-i\beta_1 - \beta_2 e^{i\theta})\rangle_6. \qquad (1)$$

The additional phase shift θ was introduced in Ref. [4] in order to smooth out the phase distribution obtained for weak fields as we discuss below.

The joint count probability for the differences $n_{43} = n_4 - n_3$ and $n_{65} = n_6 - n_5$ in the photocount distribution is

$$W(n_{43}, n_{65}|\, e^{i\theta}) = \sum_{n_3,n_4,n_5,n_6}{}' \quad |\langle n_3, n_4, n_5, n_6|\psi_{\rm out}\rangle|^2, \qquad (2)$$

where summations denoted by $\sum'$ are performed for fixed differences n_{43} and n_{65}. These summations have been previously computed [6] and we find

$$W(n_{43}, n_{65}|\, e^{i\theta}) = W_3(n_{43}|\, e^{i\theta}) W_5(n_{65}|\, e^{i\theta}), \qquad (3)$$

with

$$W_3(n_{43}|\, e^{i\theta}) \quad = \quad e^{-\frac{1}{2}(|\beta_1|^2+|\beta_2|^2)} \left|\frac{\beta_1 + \beta^2 e^{i\theta}}{\beta_1 - \beta^2 e^{i\theta}}\right|^{n_{43}} I_{|n_{43}|}\left(\tfrac{1}{2}|\beta_1^2 - \beta_2^2 e^{2i\theta}|\right), \qquad (4)$$

and

$$W_5(n_{65}|\, e^{i\theta}) \quad = \quad e^{-\frac{1}{2}(|\beta_1|^2+|\beta_2|^2)} \left|\frac{i\beta_1 + \beta^2 e^{i\theta}}{i\beta_1 - \beta^2 e^{i\theta}}\right|^{n_{65}} I_{|n_{65}|}\left(\tfrac{1}{2}|\beta_1^2 + \beta_2^2 e^{2i\theta}|\right). \qquad (5)$$

Here $I_\nu(z)$ denotes the modified Bessel function of order ν.

The (n_{43}, n_{65}) pairs are the experimental data and by construction represent $(x, p) = (r\cos\varphi, r\sin\varphi)$ pairs on phase space. Since the data is discrete it maps to a regular lattice of points on phase space. This would correspond to a 'spiky' distribution in φ. However, Ref. [4] introduced a clever trick to smooth out this distribution: Shifting field β_2 by θ, associating each pair of data with $(x_\theta, p_\theta) = (r\cos(\varphi-\theta), r\sin(\varphi-\theta))$, and finally averaging over θ. For each (n_{43}, n_{65}) pair this yields

$$\begin{aligned} \overline{W}_{n_{43},n_{65}}(x,p) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\, W_3(x\cos\theta - p\sin\theta|\, e^{i\theta})\,\delta(x\cos\theta - p\sin\theta - n_{43}) \\ &\quad \times W_5(p\cos\theta + x\sin\theta|\, e^{i\theta})\,\delta(p\cos\theta + x\sin\theta - n_{65}) \\ &= \frac{1}{\pi} W_3(n_{43}|\,(n_{43}+in_{65})(x-ip)/r^2) \\ &\quad \times W_5(n_{65}|\,(n_{43}+in_{65})(x-ip)/r^2)\delta(n_{43}^2+n_{65}^2-r^2). \end{aligned} \qquad (6)$$

Summing over all possible sets of data and integrating out the radial variable yields the general expression for the phase distribution

$$\begin{aligned} P(\varphi) &= \sum_{n_{43},n_{65}} \int_0^\infty r\,dr\, \overline{W}_{n_{43},n_{65}}(r\cos\varphi, r\sin\varphi) \\ &= \frac{1}{2\pi}\sum_{n_{43},n_{65}} W_3\left(n_{43}|\,(n_{43}+in_{65})(n_{43}^2+n_{65}^2)^{-1/2}e^{-i\varphi}\right) \\ &\quad \times W_5\left(n_{65}|\,(n_{43}+in_{65})(n_{43}^2+n_{65}^2)^{-1/2}e^{-i\varphi}\right), \end{aligned} \qquad (7)$$

where the distributions W_3 and W_5 follow from Eqs. (4) and (5).

So far the expression Eq. (7) for the phase distribution is exact. In order to gain some insight, and in particular to compare the two schemes we now consider two limiting cases.

A.1. *Strong field limit*

When one of the fields, say β_2, is strong that is $1 \ll |\beta_2|$ and $|\beta_1| \ll |\beta_2|$, we can use the known result for the strong local oscillator limit [6], where β_2 plays the role of the local oscillator. In this case, the phase distribution reduces to

$$\begin{aligned} P(\varphi) &= \frac{1}{\pi}\int_0^\infty r\,dr\, \exp\Big\{-\left[r\cos\varphi - |\beta_1|\cos(\phi_2-\phi_1)\right]^2 \\ &\quad - \left[r\sin\varphi - |\beta_1|\sin(\phi_2-\phi_1)\right]^2\Big\}, \end{aligned} \qquad (8)$$

where $\beta_j = |\beta_j|e^{i\phi_j}$, for $j = 1, 2$.

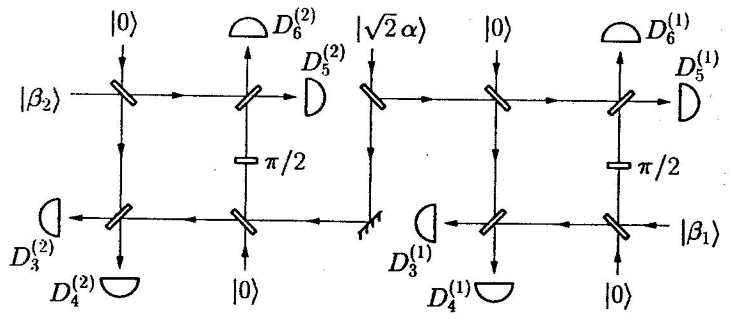

Fig. 2. Indirect measurement scheme: A pair of eight-port homodyne detectors is used to measure the phase difference between the optical fields β_1 and β_2. Each field is beat against a common strong local oscilator α. Each eight-port detector has four detectors measuring the photon count differences n_{43} and n_{65}. Two phase distributions are obtained and combined to give the phase distribution for the phase difference.

A.2. Weak fields limit

When both fields are weak, that is $|\beta_1| \ll 1$, and $|\beta_2| \ll 1$, the summation in Eq. (7) has only contributions from those terms in the lowest order in the coherent states amplitudes $|\beta_j|$. Hence only terms with $n_{43} = -1, 0, 1$ and $n_{65} = -1, 0, 1$ contribute and we arrive at

$$P(\varphi) \simeq \frac{1}{2\pi}\Big[1 + 2|\beta_1||\beta_2|\cos(\varphi + \phi_1 - \phi_2)\Big], \tag{9}$$

up to corrections which are quartic in $|\beta_j|$. Here we have used the relation

$$I_\nu(z) \simeq \frac{1}{\nu!}\left(\frac{z}{2}\right)^\nu .$$

This approach keeps all data—even when it is ambiguous. By contrast, Mandel and coworkers have consistently recommended discarding such data in their approach to defining an operational phase operator. Indeed, the distribution of Eq. (7) may be used to define a POM for the operational phase even with the ambiguous data included [7].

If the ambiguous data, having $n_{43} = n_{65} = 0$, are discarded the phase distribution becomes

$$P(\varphi) \simeq \frac{1}{2\pi}\left(1 + \frac{2\cos(\varphi + \phi_1 - \phi_2)}{|\beta_1|/|\beta_2| + |\beta_2|/|\beta_1|}\right), \tag{10}$$

to lowest order, which is exactly the result obtained by Torgerson and Mandel [1].

B. Indirect measurement

The indirect measurement scheme consists of two eight-port homodyne detectors, each performing a measument of the phase distribution for one of the coherent fields relative to a common strong local oscillator, α, as seen in Fig. 2. Since the phase distributions are completely independent from each other, we can write the joint distribution as

$$P(\varphi_1, \varphi_2) = P_1(\varphi_1)P_2(\varphi_2). \tag{11}$$

The distribution for the phase difference $\varphi = \varphi_1 - \varphi_2$ is the convolution

$$P(\varphi) = \int_0^{2\pi} d\varphi_1\, P_1(\varphi_1), P_2(\varphi_1 - \varphi) \tag{12}$$

where

$$P_i(\varphi_i) = \frac{1}{\pi}\int_0^\infty r_i\, dr_i\, \exp\Big\{-\big[r_i \cos\varphi_i - |\beta_i|\cos(\phi_0 - \phi_i)\big]^2 - \big[r_i \sin\varphi_i - |\beta_i|\sin(\phi_0 - \phi_i)\big]^2\Big\} \tag{13}$$

is the phase distribution in the strong local oscillator limit, Eq. (8), for each field and ϕ_0 is the phase of the local oscillator. Evaluating the integral in Eq.(12) we obtain the expression

$$P(\varphi) = \frac{2}{\pi}e^{-|\beta_1|^2-|\beta_2|^2}\int_0^\infty r_1\, dr_1 \int_0^\infty r_2\, dr_2\; e^{-r_1^2-r_2^2} I_0\big(2|r_1\beta_1 + r_2\beta_2 e^{-i\varphi}|\big) \tag{14}$$

for the phase distribution of the indirect measurement. Again we now consider the limiting cases of this exact expression.

B.1. Strong field limit

In the case when β_2 is strong but still much weaker than the local oscillator, that is $1 \ll |\beta_2| \ll \alpha$ and $|\beta_1| \ll |\beta_2|$, Eq. (14) approaches the result Eq. (8) for the direct measurement very quickly. This behaviour can be seen from Fig. 3, where we have plotted both expressions for $|\beta_2/\beta_1| = 4$. For higher values of $|\beta_2|$ the two curves coincide.

B.2. Weak fields limit

In the case when both fields are weak, that is when $|\beta_1| \ll 1 \ll \alpha$ and $|\beta_2| \ll 1 \ll \alpha$, the integral in Eq. (14) is easily performed and yields

$$P(\varphi) \simeq \frac{1}{2\pi}\left[1 + \frac{\pi}{2}|\beta_1||\beta_2|\cos(\varphi + \phi_1 - \phi_2)\right] \tag{15}$$

to lowest order for the phase distribution.

We conclude this section by noting that, in contrast to the direct measurement, in this indirect measurement the contribution from ambiguous data is always negligible due to the presence of the strong local oscillator.

3. To post-select or not?

In Section 2A we have discussed two possibilities of data analysis and have shown the phase distributions corresponding to retaining or discarding ambiguous data. In this section we briefly return to this point.

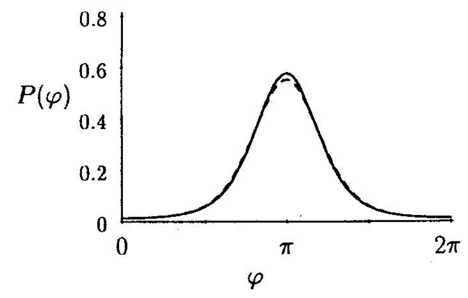

Fig. 3. Comparison between phase distributions for both direct (solid line) and indirect measurement (dashed line), in the strong field limit. In this limit, the phase distribution of the direct scheme follows from Eq. (8), whereas for the indirect measurement, we have used the general expression Eq. (14). For both curves we have chosen $\beta_1 = 1$ and $\beta_2 = -4$.

The data analysis used in Eq. (6) may be compactly summarized by considering the distribution for the random variable $Z = f(X,Y)$ given the distribution $P(X,Y)$ for the random variables X and Y. This can be written as

$$P(Z) \propto \int dX dY \delta(Z - f(X,Y)) P(X,Y).$$

When the data, which in this case is a pair (X, Y), unambiguously determines Z the above formula is trivial; however, when the data leads to no single Z the above result spreads the probability evenly amongst the consistent values.

In contrast the approach used in Ref. [4] discards data obtained for $n_{43} = n_{65} = 0$. The excluded data does not appear to be very useful for informing us about phase. Thus, it is justifiable to select only the unambiguous data as has been done in Ref. [4]. However, a comparison of the behavior of the direct scheme without and with postselection for weak fields, Eqs. (9) and (10) respectively, shows that the latter has a significantly narrower distribution. How can we reconcile this with Shannon's information theory which teaches us that we cannot improve sensitivity by discarding information? It must be that the apparent difference in widths—and naively sensitivity—for these two distributions is in some sense illusory. The resolution to this 'paradox' is that the discarded data carries information about the overall resources used which would need to be factored into any meaningful measure of sensitivity. The deeper question of how we may compare phase distributions between different schemes, with respect to the cost of resources involved, is beyond the scope of this paper.

4. Conclusion

We may generate the distribution for the relative phases of a pair of coherent states β_1 and β_2 using the direct or indirect schemes described here. When at least one of the fields is strong these schemes are indistinguishable. By contrast, for weak fields these schemes lead to slightly different distributions. Retaining or discarding ambiguous data makes no difference in the schemes' sensitivity. Careful comparisons between these schemes can only be performed in the presence of the costs associated with running them. Notwithstanding this caution, our results confirm Torgerson and Mandel's claim of the non-uniqueness of operational phase operators.

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