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QUANTUM PHASE MEASUREMENT BY BALANCED HOMODYNING[1]

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The problem of measuring the London phase state distribution and the Susskind–Glogower cosine and sine phase state distributions of a radiation-field mode in balanced homodyning is studied. Appropriately smoothed phase state distributions are introduced that tend to the exact ones as the smoothing parameters approach zero. The integral relations of the smoothed phase state distributions to the phase-parametrized field-strength distributions measurable in balanced homodyning are studied. It is shown that for nonzero smoothing parameters the integral kernels are well-behaved functions. The integral relations can therefore be applied to direct sampling of the smoothed phase state distributions from the field-strength distributions. This offers the possibility of asymptotic determination of the exact phase state distributions from the difference-count statistics recorded in balanced homodyning in a very direct way. Numerical simulations show that the method yields the exact phase distributions with sufficient accuracy.

1. Introduction

The quantum-mechanical description of amplitude and phase quantities and their measurement has turned out to be troublesome and is still a matter of discussion (for a review, see [1]). Since Dirac's work in 1927 [2] a number of attempts have been made to define phase operators. The main reason for the difficulties connected with a definition of phase in quantum theory has been the lack of a unique self-adjoint phase operator. So, various phase concepts have been developed that converge in the classical limit but give quite different insight in the problem of quantum phase.

In this context the question has been arisen of how the different phase quantities introduced in quantum theory can be measured – a question which is of particular interest with regard to phase definitions that are closely related to phase operators. A

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powerful method of measurement of phase sensitive properties of radiation-field modes has been homodyne and heterodyne detection techniques. Moreover, these techniques have offered qualitatively new possibilities of measuring the quantum state of the mode [3,4].

In the following we study the problem of measuring the London phase state distribution and the Susskind–Glogower cosine and sine phase state distributions of a radiation-field mode by direct sampling them from the recorded data in balanced homodyning. For this purpose we introduce appropriately smoothed phase state distributions, where the smoothing parameters characterize the observational level at which the phase state distributions are desired to be observed. The exact distributions are asymptotically observed in the limit when the smoothing parameters approach zero. To illustrate the method, results of computer simulations are presented.

2. Parametrized phase state distributions

Let us restrict attention to a single-mode radiation field prepared in a quantum state described by a density operator $\hat{\varrho}$ and consider the exponential phase operator

$$\hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \tag{1}$$

and the Hermitian cosine and sine operators

$$\hat{C} = \frac{1}{2}(\hat{E} + \hat{E}^{\dagger}) \quad \text{and} \quad \hat{S} = \frac{1}{2i}(\hat{E} - \hat{E}^{\dagger}), \tag{2}$$

respectively, which satisfy the eigenvalue equations [5,6]

$$\hat{E}\,|e^{i\phi}\rangle = e^{i\phi}\,|e^{i\phi}\rangle, \tag{3}$$

$$\hat{C}\,|\cos\phi\rangle = \cos\phi\,|\cos\phi\rangle, \quad \hat{S}\,|\sin\phi\rangle = \sin\phi\,|\sin\phi\rangle. \tag{4}$$

The London phase states $|e^{i\phi}\rangle$ as well as the Susskind–Glogower cosine and sine phase states $|\cos\phi\rangle$ and $|\sin\phi\rangle$, respectively, resolve the unity. The states $|e^{i\phi}\rangle$ are non-orthogonal and non-normalizable. Both the cosine states $|\cos\phi\rangle$ and the sine states $|\sin\phi\rangle$ are orthonormal with respect to the Dirac δ function. The states can be used to define quantum phase distributions of the mode under consideration.

With regard to direct sampling of the distributions from the data recorded in balanced homodyning, it is useful to start from proper Hilbert space states $|\phi,\epsilon\rangle \equiv |e^{i\phi},\epsilon\rangle$, $|\cos\phi,\epsilon\rangle$, and $|\sin\phi,\epsilon\rangle$ that are normalized to unity,

$$|\phi,\epsilon\rangle = (1-e^{-2\epsilon})^{1/2}\sum_{n=0}^{\infty} e^{-n\epsilon}e^{in\phi}|n\rangle, \tag{5}$$

$$|\cos\phi,\epsilon\rangle = \frac{1}{\sqrt{\epsilon}}\int_{\phi-\epsilon/2}^{\phi+\epsilon/2} \mathrm{d}\phi'\,|\cos\phi'\rangle, \quad |\sin\phi,\epsilon\rangle = \frac{1}{\sqrt{\epsilon}}\int_{\phi-\epsilon/2}^{\phi+\epsilon/2} \mathrm{d}\phi'\,|\sin\phi'\rangle. \tag{6}$$

The non-orthogonal states $|\phi,\epsilon\rangle$ satisfying the eigenvalue equation $\hat{E}|\phi,\epsilon\rangle = e^{-\epsilon}e^{i\phi}|\phi,\epsilon\rangle$ are also called coherent phase states (see, e.g., [7]). After multiplication by the factor $(1-e^{-2\epsilon})^{-1/2}$ they tend to the non-normalizable London phase states $|e^{i\phi}\rangle$ as ϵ approaches zero. Similarly, the states $|\cos\phi\rangle$ and $|\sin\phi\rangle$, respectively, are obtained from the states $\epsilon^{-1/2}|\cos\phi,\epsilon\rangle$ and $\epsilon^{-1/2}|\sin\phi,\epsilon\rangle$ in the limit when $\epsilon \to 0$.

Using the states in Eqs. (5) and (6), ϵ-parametrized phase distributions can be introduced via their overlaps with the quantum state of the radiation-field mode under consideration,

$$p(\phi,\epsilon) = N^{-1}(\epsilon)\,\langle\phi,\epsilon|\,\hat{\varrho}\,|\phi,\epsilon\rangle, \tag{7}$$

$$p_{\rm c}(\phi,\epsilon) = N_{\rm c}^{-1}(\epsilon)\,\langle\cos\phi,\epsilon|\,\hat{\varrho}\,|\cos\phi,\epsilon\rangle, \tag{8}$$

$$p_{\rm s}(\phi,\epsilon) = N_{\rm s}^{-1}(\epsilon)\,\langle\sin\phi,\epsilon|\,\hat{\varrho}\,|\sin\phi,\epsilon\rangle, \tag{9}$$

where the normalization factors $N(\epsilon)$, $N_{\rm c}(\epsilon)$, and $N_{\rm s}(\epsilon)$ have been introduced in order to normalize the distributions to unity. The distributions (7) – (9) are smoothed phase distributions (cf. the Figure) which in the limit $\epsilon \to 0$ become the London phase state distribution and the Susskind–Glogower cosine and sine phase distributions [8]. The smoothing parameter ϵ can be regarded as being a measure of the observational level at which the exact distributions are desired to be obtained. To answer the question of whether or not the phase distributions can directly be sampled from the data recorded in balanced homodyning, let us now consider their relations to the phase-parametrized field-strength distributions $p(\mathcal{F},\varphi) = \langle\mathcal{F},\varphi|\,\hat{\varrho}\,|\mathcal{F},\varphi\rangle$, where $|\mathcal{F},\varphi\rangle$ are the eigenkets of single-mode field strengths $\hat{F}(\varphi) = |F|(e^{-i\varphi}\hat{a} + e^{i\varphi}\hat{a}^{\dagger})$ [$\hat{a}$ ($\hat{a}^{\dagger}$), photon destruction (creation) operator]; for details see, e.g., [9].

3. Relation to the field-strength distributions

It is well known that in perfect homodyne detection the measured difference-count statistics is an appropriately scaled field-strength distribution $p(\mathcal{F},\varphi)$ [10]. Knowing the field-strength distributions for all values of φ within a π interval, the density operator $\hat{\varrho}$ is also known, namely [4,11]

$$\hat{\varrho} = \int_0^{\pi} d\varphi \int d\mathcal{F}\, p(\mathcal{F},\varphi)\,\hat{K}(\mathcal{F},\varphi), \tag{10}$$

where the operator integral kernel $\hat{K}(\mathcal{F},\varphi)$ is given by

$$\hat{K}(\mathcal{F},\varphi) = \frac{|F|^2}{\pi}\int dy\,|y|\,\exp\Big\{iy\Big[\hat{F}(\varphi) - \mathcal{F}\Big]\Big\}. \tag{11}$$

From Eqs. (10) and (7) – (9) we find that

$$p(\phi,\epsilon) = N^{-1}(\epsilon)\int_0^{\pi} d\varphi \int_{-\infty}^{\infty} d\mathcal{F}\, p(\mathcal{F},\varphi)\, K_{\epsilon}(\phi,\mathcal{F},\varphi) \tag{12}$$

and

$$p_{\rm c(s)}(\phi,\epsilon) = N_{\rm c(s)}^{-1}(\epsilon)\int_0^{\pi} d\varphi \int_{-\infty}^{\infty} d\mathcal{F}\, p(\mathcal{F},\varphi)\, K_{\epsilon}^{\rm c(s)}(\phi,\mathcal{F},\varphi). \tag{13}$$

Using Eqs. (5) and (6) and representing the cosine and sine phase states $|\cos\phi\rangle$ and $|\sin\phi\rangle$, respectively, in the photon-number basis (see, e.g., [5]), the integral kernels can be written as

$$K_\epsilon(\phi,\mathcal{F},\varphi) = \langle\phi,\epsilon|\,\hat{K}(\mathcal{F},\varphi)\,|\phi,\epsilon\rangle = (1-e^{-2\epsilon})\sum_{n,m} K_{nm}(\mathcal{F},\varphi)\, e^{i(n-m)\phi} e^{-\epsilon(n+m)}, \quad (14)$$

$$K^{c}_\epsilon(\phi,\mathcal{F},\varphi) = \langle\cos\phi,\epsilon|\,\hat{K}(\mathcal{F},\varphi)\,|\cos\phi,\epsilon\rangle = \frac{8}{\pi\epsilon}\sum_{n,m}\left\{\frac{K_{nm}(\mathcal{F},\varphi)}{(n+1)(m+1)}\sin[(n+1)\phi]\,\sin[(m+1)\phi] \times \sin[(n+1)\epsilon/2]\,\sin[(m+1)\epsilon/2]\right\}, \quad (15)$$

$$K^{s}_\epsilon(\phi,\mathcal{F},\varphi) = \langle\sin\phi,\epsilon|\,\hat{K}(\mathcal{F},\varphi)\,|\sin\phi,\epsilon\rangle = \frac{8}{\pi\epsilon}\sum_{n,m}\left\{\frac{K_{2n\,2m}(\mathcal{F},\varphi)}{(2n+1)(2m+1)}\cos[(2n+1)\phi]\,\cos[(2m+1)\phi] \times \sin[(2n+1)\epsilon/2]\,\sin[(2m+1)\epsilon/2] + \frac{K_{2n+1\,2m+1}(\mathcal{F},\varphi)}{4(n+1)(m+1)}\sin[2(n+1)\phi]\,\sin[2(m+1)\phi]\,\sin[(n+1)\epsilon]\,\sin[(m+1)\epsilon]\right\}. \quad (16)$$

Here,

$$K_{nm}(\mathcal{F},\varphi) = \langle n|\,\hat{K}(\mathcal{F},\varphi)\,|m\rangle \quad (17)$$

is the sampling function for determining the density matrix in the photon-number basis, which has been studied in detail in a number of papers; see, e.g., [12,13]. It can be given by

$$K_{nm}(\mathcal{F},\varphi) = e^{i(n-m)\varphi} f_{nm}(x), \quad (18)$$

where $f_{nm}(x)$ may be expressed in terms of parabolic cylinder functions, x being the dimensionless variable $\mathcal{F}/(\sqrt{2}|F|)$. It should be noted that in the case of non-perfect detection smeared field-strength distributions are measured that are convolutions of the true distributions $p(\mathcal{F},\varphi)$ with a Gaussian noise distribution; see, e.g., [9]. Substituting in Eqs. (12) and (13) for $p(\mathcal{F},\varphi)$ the smeared distributions, in Eqs. (14) – (16) the functions $K_{nm}(\mathcal{F},\varphi)$ must be modified accordingly; see, e.g., [12,13].

From inspection of Eqs. (14) – (16) we find that for any $\epsilon>0$ the functions $K_\epsilon(\phi,\mathcal{F},\varphi)$ and $K^{c(s)}_\epsilon(\phi,\mathcal{F},\varphi)$ are well-behaved (bounded) functions. Hence, Eqs. (12) and (13) can be regarded as basic equations for direct sampling of $p(\phi,\epsilon)$, $p_c(\phi,\epsilon)$, and $p_s(\phi,\epsilon)$ from $p(\mathcal{F},\varphi)$ for any $\epsilon>0$. Since the sampling functions do not exist for $\epsilon=0$ (there is no convergence in the Fock basis expansions in this case), the exact London phase state distribution and the exact Susskind–Glogower cosine and sine phase state distributions can only be obtained asymptotically. The smaller the value of ϵ is, the higher the level of accuracy becomes at which the exact distributions can be obtained.

4. Computer simulation of measurements

To demonstrate the feasibility of direct sampling of the phase distributions (at chosen observation level) from the difference-count statistics recorded in perfect homodyne detection, we have performed computer simulations for a signal-field mode prepared in a squeezed vacuum state with a mean number of photons $\langle\hat{n}\rangle=1$. In particular, we have assumed that $10^4\times 30$ events are recorded. From the results in the Figure 1. we see that the sampled distributions are in good agreement with the theoretical predictions. Their deviations from the exact distributions obviously result from the middle observational level chosen. Nevertheless, the accuracy is seen to be sufficient in order to detect the typical features of the exact distributions.

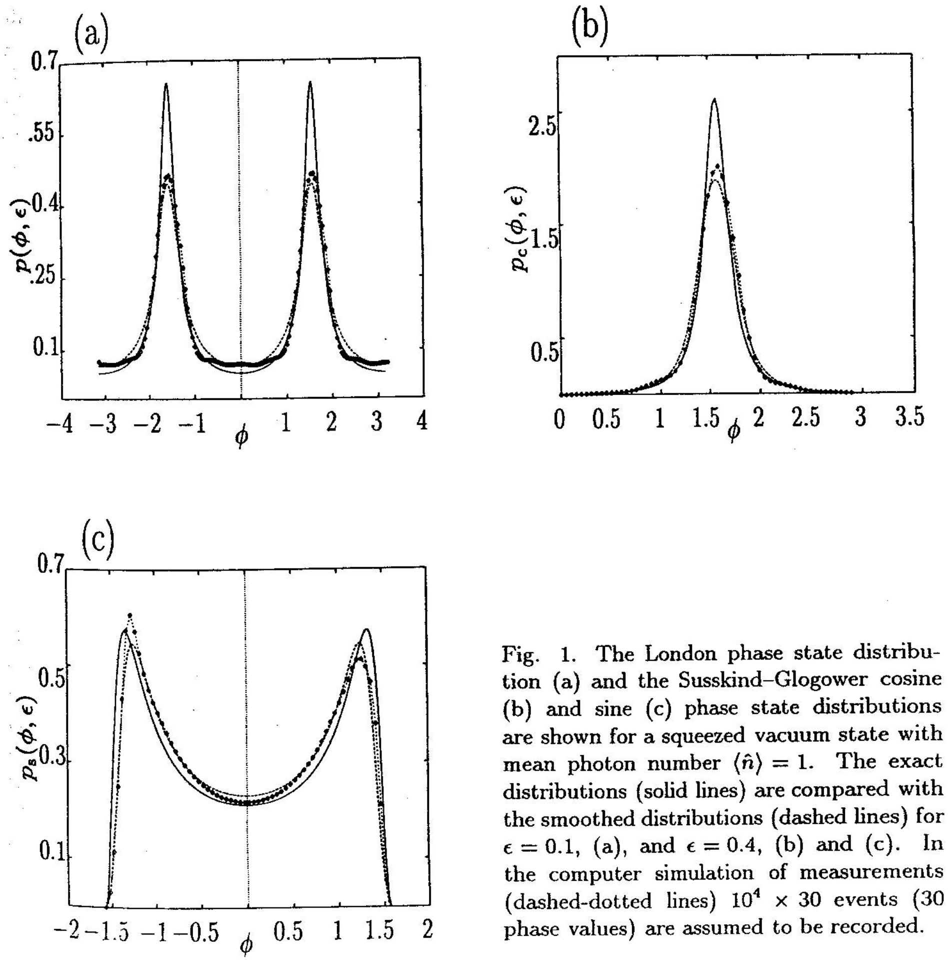

Fig. 1. The London phase state distribution (a) and the Susskind–Glogower cosine (b) and sine (c) phase state distributions are shown for a squeezed vacuum state with mean photon number $\langle\hat{n}\rangle = 1$. The exact distributions (solid lines) are compared with the smoothed distributions (dashed lines) for $\epsilon = 0.1$, (a), and $\epsilon = 0.4$, (b) and (c). In the computer simulation of measurements (dashed-dotted lines) $10^4\times 30$ events (30 phase values) are assumed to be recorded.

5. Conclusions

Introducing appropriately smoothed phase distributions, we have shown that both the London phase state distribution and the Susskind–Glogower cosine and sine phase state distributions of a single-mode radiation field can directly be sampled from the recorded difference-count statistics in balanced homodyning with sufficiently well accuracy. The accuracy is determined by the interval of field strengths used for "probing" the phase statistics of the state under consideration. With decreasing smoothing parameter this interval is increased and hence, the accuracy is increased as well. Since for vanishing smoothing parameter the sampling functions do not exist, the exact phase distributions can only be obtained asymptotically.

In the paper we have calculated the sampling functions using expansions in the photon-number basis. Since with decreasing value of the smoothing parameter the values of the photon numbers which must be taken into account increase, highly oscillating functions are involved in the expansions and much numerical effort must be made to calculate the sampling functions. This difficulty might be overcome by using consequently a field-strength basis [14] and avoiding the detour via the photon-number basis, as has recently been demonstrated for direct sampling of the density matrix in a field-strength basis [15].

We finally note that the results reveal that the above considered quantum phase distributions cannot be obtained, in general, from the field-strength distributions in the sense of an operational phase distribution introduced in [16].

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