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JOINT PHASE SPACE DESCRIPTION OF ATOM–FIELD INTERACTIONS[1]

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We define a joint Wigner function to give a fully phase space description for the interacting system of the quantized electromagnetic field and two-level atoms. We give the equation of motion in a special case and present a simple solution.

1. Introduction

An atom interacting nearly resonantly with a single mode electromagnetic field can be approximated with good accuracy as a two-level system, that is as a $\frac{1}{2}$-spin particle. A fully quantum-theoretical description of this system is the Jaynes–Cummings model [1]. If one has a collection of N identical two-level atoms, then the behaviour of the atoms is similar to the dynamics of an agular momentum characterised by the quantum number $j = N/2$ [2, 3]. This model is the prototype of superradiance [2]. The state space of these kind of interacting atom–field systems is the direct product of the $(2j+1)$-dimensional angular momentum state space with the infinite-dimensional oscillator state space.

It is customary to apply phase space methods to the oscillator field mode [4, 5], while the same technique is much less exploited for the atomic system. The first quasiprobability distribution for the spin variable over a sphere as an appropriate phase space is due to Stratonovich [6]. The idea that the coherent quantum states of an ensemble of two-level atoms can be associated with points on a surface of a sphere has been first discussed by Arecchi & al. [3]. The usage of atomic phase space distributions in quantum optics has been demonstrated more recently in [7, 8].

Now it seems straightforward to describe the interaction of the atomic system and the field by a new quasi-probability distribution function over the corresponding (classical) phase space, that is over the direct product of a spherical surface with a plane.

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We introduce here the joint Wigner function of the interacting atom–field system to give a fully phase space description for these typical problems of quantum optics. Our definition is based on unifying the well known Wigner functions [9, 10] for the field oscillator [4, 5] and for the angular momentum [6, 7]. We give the equation of motion for the joint Wigner function in a special case and give its solution assuming a simple initial condition.

2. Comparison of the Wigner functions for field and for angular momentum

To see the analogy between the definitions of the Winger functions for oscillator and for angular momentum we briefly summarize both of them below.

It is customary to define the Wigner function of a field mode in terms of the (symmetrically ordered) characteristic function:

$$\chi(\beta,\beta^*) = \mathrm{Tr}(\rho D(\beta,\beta^*)), \tag{1}$$

where ρ is the density operator of the field and $D(\beta,\beta^*) = e^{\beta a^\dagger - \beta^* a}$ is the usual displacement operator.

The Wigner fuction is simply the Fourier-transform of the characteristic function:

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\beta\, \chi(\beta,\beta^*) e^{\alpha\beta^* - \alpha^*\beta} \tag{2}$$

$$= \mathrm{Tr}\left(\rho\, \frac{1}{\pi^2}\int d^2\beta\, D(\beta,\beta^*) e^{\alpha\beta^* - \alpha^*\beta}\right), \tag{3}$$

where we combined (2) with (1) to obtain (3). We note that the Fourier transform of $D(\beta,\beta^*)$ appearing in (3) is just the operator of the parity with respect of the phase space point α, and it follows that the Wigner function is the classical counterpart of this displaced parity operator [12].

In order to get a similar construction in the case of an angular momentum of quantum number j one introduces the phase space for this system, which is a sphere of radius $\hbar\sqrt{j(j+1)}$. The natural complete orthonormal function basis on this sphere is the set of the spherical harmonics $Y_{KQ}(\theta,\phi)$, while the corresponding operator basis is constituted by the $(2j+1)^2$ irreducible multipole tensor operators T_{KQ} [13]. The latter satisfy by definition the following commutation relations :

$$[J_z, T_{KQ}] = \hbar Q T_{KQ}, \tag{4}$$

$$[J_\pm, T_{KQ}] = \hbar\sqrt{K(K+1) - Q(Q\pm 1)}\, T_{K,Q\pm 1}, \tag{5}$$

where J_z, J_+, J_- are the usual components of the total angular momentum for the corresponding system. The K indices are nonegative integers and $Q = -K, -K+1, \ldots, K$. This operator basis is complete in the sense that any operator acting in the $(2j+1)$-dimensional angular momentum state space can be expanded in terms of these operators. Further, the T_{KQ}-s are orthonormal with respect to the Hilbert-Schmidt norm:

$$\mathrm{Tr}(T^\dagger_{KQ} T_{K'Q'}) = \delta_{KK'}\delta_{QQ'}. \tag{6}$$

The angular momentum Wigner function is defined then for the density operator ρ in terms of the quantities $\rho_{KQ} = \mathrm{Tr}(\rho T^\dagger_{KQ})$ which are the counterparts of the characteristic function (1). The Wigner function itself is the discrete Fourier-type transform of the ρ_{KQ}-s [6, 7]:

$$W(\theta,\phi) = \sqrt{\frac{2j+1}{4\pi}} \sum_{K=0}^{2j} \sum_{Q=-K}^{K} \rho_{KQ} Y_{KQ}(\theta,\phi). \tag{7}$$

3. Definition of the joint Wigner function

Now we introduce the joint Wigner function for interacting atom–field systems. This new quasi-probability distribution function is defined over the corresponding classical phase space, which is the direct product of a spherical surface characterized by the coordinates (θ,ϕ) and the (α,α^*)-plane.

As usual in connection with quasi-probability density functions, we introduce first the joint characteristic function in the following way:

$$C_{KQ}(\beta,\beta^*,t) = \mathrm{Tr}\left(\rho(t)\, D(\beta,\beta^*)\, T^\dagger_{KQ}\right), \tag{8}$$

where $\rho(t)$ is the density operator of the interacting system, $D(\beta,\beta^*)$ is the displacement operator acting in the field's state space, and T_{KQ} is the multipole operator acting in the angular momentum state space. The tensor product operators $D(\beta,\beta^*)\,T^\dagger_{KQ}$ form the basis (in the space of operators acting in the state space of the interacting system) which is appropriate for the definition of the joint Wigner function.

With the help of the joint characteristic function we define the joint Wigner function as

$$W(\alpha,\alpha^*,\theta,\phi,t) = \sqrt{\frac{2j+1}{4\pi^5}} \sum_{K=0}^{2j} \sum_{Q=-K}^{K} Y_{KQ}(\theta,\phi) \int d^2\beta\, C_{KQ}(\beta,\beta^*,t) e^{\alpha\beta^* - \alpha^*\beta}. \tag{9}$$

We note that by a definition similar to (9) we can map any operator A representing an observable onto a phase space function $\mathcal{A}(\alpha,\alpha^*,\theta,\phi,t)$, which enables the calculation of quantum expectation values by integrations over the phase space. This way one arrives at the Wigner representation of these interacting atom–field systems.

4. Equation of motion

Now we give the equation of motion for the joint Wigner function, assuming the special case of $j = \frac{1}{2}$, i.e. one two-level atom. We describe the interaction by the Jaynes–Cummings model, that is by the Hamilton operator

$$H_{JC} = \hbar\omega_f(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\hbar\omega_a\sigma_z + \hbar g(a\sigma_+ + a^\dagger\sigma_-). \tag{10}$$

By using the dynamical equation for ρ we find the following equation of motion for the joint Wigner function:

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W} &= -\omega_f\frac{\partial}{\partial\psi}\mathcal{W}-\omega_a\frac{\partial}{\partial\phi}\mathcal{W}\\ &+2g|\alpha|\sin(\phi-\psi)\frac{\partial}{\partial\theta}\mathcal{W}+2g|\alpha|\cot\theta\cos(\phi-\psi)\frac{\partial}{\partial\phi}\mathcal{W}\\ &+\frac{\sqrt{3}}{2}g\cos^2\theta\frac{\partial}{\partial\theta}\frac{1}{\cos\theta}\left\{\left(\sin(\phi-\psi)+\cos(\phi-\psi)\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial|\alpha|}\right.\\ &\left.-\frac{1}{|\alpha|}\left(\cos(\phi-\psi)-\sin(\phi-\psi)\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\psi}\right\}\mathcal{W}\\ &-g\frac{1}{\sqrt{3}\sin\theta}\left\{\left(\cos(\phi-\psi)-\sin(\phi-\psi)\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\phi}\frac{\partial}{\partial|\alpha|}\right.\\ &\left.+\frac{1}{|\alpha|}\left(\sin(\phi-\psi)+\cos(\phi-\psi)\frac{\partial}{\partial\phi}\right)\frac{\partial}{\partial\phi}\frac{\partial}{\partial\psi}\right\}\mathcal{W},\end{aligned} \tag{11}$$

where we have introduced the notation $\alpha=|\alpha|e^{-i\psi}$. This form explicitly shows the importance of the relative phases of the atom and the field.

The first and second terms in the right hand side of the equation of motion describe the time evolution of the Wigner function in the absence of interaction, that is the dynamics of the free field and the free atom. In this case the solution of (11) does not need any additional effort compared to the equation of motion for the field's usual Wigner function.

However, it is a more difficult task to solve (11) in the case of interaction. If initially the atom is in its upper state and the field is in the n-th number state, then the solution of (11) is the following:

$$\begin{aligned}\mathcal{W}_n^+(\alpha,\alpha^*,\theta,\phi,t) &= \frac{(-1)^n}{2\pi^2}e^{-2|\alpha|^2}\left\{(1+\sqrt{3}\cos\theta)|d_n^+(t)|^2L_n^{(0)}(4|\alpha|^2)\right.\\ &+4\sqrt{\frac{3}{n+1}}\sin\theta\,L_n^{(1)}(4|\alpha|^2)\,\mathrm{Re}\left[e^{i\phi}\left(d_{n+1}^-(t)\right)^*d_n^+(t)\,\alpha\right]\\ &\left.+(\sqrt{3}\cos\theta-1)|d_{n+1}^-(t)|^2L_{n+1}^{(0)}(4|\alpha|^2)\right\}.\end{aligned} \tag{12}$$

Here $L_n^{(a)}(x)$ denotes the Laguerre polynomial [14] and

$$d_n^+(t) = \cos(R_nt)+i\frac{\omega_f-\omega_a}{2R_n}\sin(R_nt), \tag{13}$$

$$d_{n+1}^-(t) = -i\frac{g\sqrt{n+1}}{R_n}\sin(R_nt), \tag{14}$$

where $R_n=\sqrt{(\omega_f-\omega_a)^2/4+g^2(n+1)}$ is the quantum Rabi frequency of the transition.

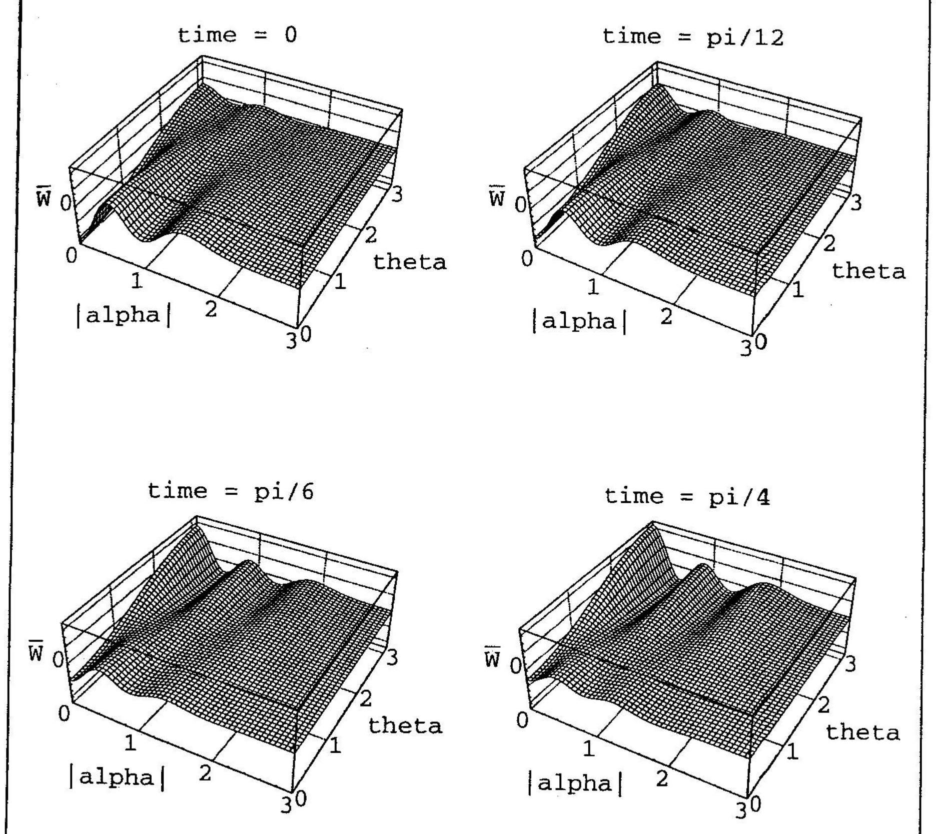

Fig. 1. Plots of the reduced joint Wigner function obtained from (12) as described in the text, assuming $n=3$, $\omega_f=\omega_a$ and $g=1$ (which correspond to a Rabi period $\pi/2$), at four different instants. At $t=0$ the atom is excited and the field is in the $n=3$ number state, while at $t=\pi/4$ the field is in the $n=4$ number state and the atom is in its ground state. At intermediate times the system is in an entangled state.

If we integrate the joint Wigner function (9) over the field variables, we obtain the usual Wigner function for the angular momentum, while integration over the spherical coordinates yields the usual Wigner function of the field.

It is more interesting however, to integrate simultaneously over one field and one atomic variable, since the resulting reduced joint Wigner function still contains information about possible correlations. For example integration over the field's phase ψ and the atomic azimuthal angle ϕ yields a reduced joint Wigner function $\overline{\mathcal{W}}(|\alpha|,\theta,t)$, which characterises simultaneously the field's intensity and the atomic inversion. With the help of this reduced joint Wigner function one can still calculate the exact expectation values of certain observables whose Wigner representation depends only on the remaining variables.

Fig. 1. shows $\overline{\mathcal{W}}_n^+(|\alpha|,\theta,t)$ obtained from (12) for $n = 3$, assuming exact resonance and $g = 1$, at four different instants.

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