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INTRINSIC AND OPERATIONAL OBSERVABLES IN QUANTUM MECHANICS[1]

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The concept of intrinsic and operational observables in quantum mechanics is introduced. It is argued that, in any realistic description of a quantum measurement that includes a detecting device, it is possible to construct from the statistics of the recorded raw data a set of operational quantities that correspond to the intrinsic quantum mechanical observable. Using the concept of the propensity and the associated operational positive operator valued measure (POVM) a general description of the operational algebra of quantum mechanical observables is derived for a wide class of realistic detection schemes. This general approach is illustrated by the example of an operational Malus measurement of the spin phases and by an analysis of the operational homodyne detection of the phase of an optical field with a squeezed vacuum in the unused ports.

1. Introduction

In the standard formulation of quantum mechanics, the statistical outcomes of an ideal measurement of an observable $\hat{A}|a\rangle = a|a\rangle$ are described by the spectral measure [1]:

$$p_\psi(a) = |\langle a|\psi\rangle|^2 \tag{1}$$

where $|\psi\rangle \in H$ is the state vector of the measured system. The spectral measure contains all the relevant statistical informations about the system, but it makes no reference to the apparatus employed in the actual measurement. Because of this property we shall refer to $\hat{A}$ as to an intrinsic quantum observable.

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A realistic experiment necessarily involves additional degrees of freedom [2] which eventually enable the experimenter to convert the raw data into a operational *propensity* density, $\Pr(a)$ of a classical variable a [3-4]. This propensity depends on the state of the system and on all the devices used in a realistic detection scheme. All these additional devices will be referred to as a filter $\mathcal{F}$, that represents the experimental setup required for the measurement of the observable $\hat{A}$. The measuring device is described by the following positive and hermitian operator $\hat{\mathcal{F}}(a)$ that satisfies the relation:

$$\int da\ \hat{\mathcal{F}}(a) = 1. \tag{2}$$

In terms of this filter operator the propensity is

$$\Pr(a) = \langle \hat{\mathcal{F}}(a) \rangle. \tag{3}$$

We see that in a realistic measurement the spectral measure in the decomposition of $\hat{A}$ is effectively replaced by a positive operator valued measure (POVM) $da\ \hat{\mathcal{F}}(a)$ [5]. In view of the linear relation between the propensity and the POVM, the operational statistical moments of the measured quantity are:

$$\overline{a^n} = \int da\, a^n \Pr(a) \ = \int da\, a^n \langle \hat{\mathcal{F}}(a) \rangle = \langle \hat{A}^{(n)}_{\mathcal{F}} \rangle \tag{4}$$

where

$$\hat{A}^{(n)}_{\mathcal{F}} = \int da\, a^n \hat{\mathcal{F}}(a)\,, \tag{5}$$

defines a unique set of operational observables associated with a given POVM for a given filter $\mathcal{F}$ [6].

As a rule, the algebraic properties of the $\hat{A}^{(n)}_{\mathcal{F}}$ operators are quite different from those of the powers of $\hat{A}$. In particular, a factorization is typically impossible, so that, for instance, $\hat{A}^{(2)}_{\mathcal{F}}$ does not equal $(\hat{A}^{(1)}_{\mathcal{F}})^2$.

It is the purpose of this work to provide an explicit construction of the POVM and the associated set of operational observables for two distinct systems, both leading to an operational algebra of sinus and cosines operators. The first system will be related to phases of the spin [7] probed by the so called Malus filter [8], and the second system will be an optical field probed by the so called homodyne filter [9].

In both cases we shall derive operational operators corresponding to the phases of the spin s or of the optical field. These operational observables will define an operational quantum trigonometry of the corresponding phase measurements.

2. Spin Operational Observables

In this section we derive operational operators of the spin phases and describe a simple idealized experimental scheme leading to such operational observables. This experiment is based on the Malus law for spin. This law predicts that the transmission of a spin-$\frac{1}{2}$ through a measuring apparatus is given by $\cos^2 \frac{\alpha}{2}$, where α is the relative angle between the orientation of the detected spin and the orientation of the Stern-Gerlach polarizer. This property can be generalized to a system with arbitrary spin s. We shall assume that such a system is described by spin coherent states $|\Omega\rangle$, where the solid angle characterizes an arbitrary spin orientation on a unit sphere. These spin s coherent states are obtained by a rotation of the maximum "down" spin state $|s,-s\rangle$ [10]:

$$|\Omega\rangle = \exp(\tau \hat{S}_+ - \tau^* \hat{S}_-)|s,-s\rangle, \tag{6}$$

where $\tau = \frac{1}{2}\theta e^{-i\phi}$ and $\hat{S}_\pm$ are the spin-s ladder operators. The spin coherent states form an over complete set of states on the Bloch sphere:

$$\frac{2s+1}{4\pi} \int d\Omega\ |\Omega\rangle\langle\Omega| = I. \tag{7}$$

Using these formulas, it is easy to obtain the Malus probability for a transmission of such a state through a Stern-Gerlach apparatus with orientation Ω'. As a result one obtains:

$$p = |\langle\Omega|\Omega'\rangle|^2 = (\cos\frac{\alpha}{2})^{4s}. \tag{8}$$

This quantum mechanical expression for the transmission function provides a generalization of the spin-$\frac{1}{2}$ Malus Law to the case of an arbitrary spin s. A measurement leading to the Malus law can be easily constructed at least in principle. Let assume that the Hilbert space of the system is extended by a filtering device (another spin-s) initially in the "down" spin state. A measurement is described by a dynamical process which generates a correlation between the system being detected and the measuring filter. Due to the unitarity of the interaction with the filter, it is possible to select the interaction parameters in such a way, that the wave function of the combined system evolves in the following way:

$$|s,-s\rangle_{\mathcal{F}} \otimes |\Omega\rangle \ \rightarrow\ |\Omega\rangle_{\mathcal{F}} \otimes |s,-s\rangle \tag{9}$$

From this relation it is clear that a measurement of the filter orientation leads to the spin Malus law, which in the space of the detected spin is equivalent to the following propensity:

$$\Pr(\Omega') = \frac{2s+1}{4\pi} |\langle\Omega'|\Omega\rangle|^2. \tag{10}$$

This relation shows that the corresponding POVM is just:

$$\hat{\mathcal{F}}(\Omega) = \frac{2s+1}{4\pi} |\Omega\rangle\langle\Omega| \ \text{ with } \int d\Omega\ \hat{\mathcal{F}}(\Omega) = 1. \tag{11}$$

Having this simple picture of spin measurement we will look for quantum operational observables connected with the Malus experiment. There is a variety of operational operators that can be associated with such phase measurements. For example the statistical moments of the azimuthal orientation are given by:

$$\overline{\cos^n\theta} = \int d\Omega\ \cos^n\theta \Pr(\Omega) \ = \langle \hat{\Theta}^{(n)}_{\mathcal{F}} \rangle \tag{12}$$

where the operational azimuthal cosine operators

$$\hat{\Theta}_{\mathcal{F}}^{(n)} = \int d\Omega\, \cos^n\theta \hat{\mathcal{F}}(\Omega)\,, \tag{13}$$

define a unique set of operational observables associated with this POVM. All integrals in this expression can be calculated and we obtain:

$$\hat{\Theta}_{\mathcal{F}}^{(n)} = \mathrm{F}(-n, s+\hat{S}_3+1, 2s+2; 2), \tag{14}$$

where $\mathrm{F}(a,b,c;z)$ is a hypergeometric function. The first two operational azimuthal cosine operators are:

$$\hat{\Theta}_{\mathcal{F}}^{(1)} = -\frac{1}{1+s}\hat{S}_3, \tag{15}$$

$$\hat{\Theta}_{\mathcal{F}}^{(2)} = \frac{2}{(1+s)(3+2s)}\hat{S}_3^2 + \hat{1}\frac{1}{3+2s}. \tag{16}$$

In the same way one can construct a corresponding set of operators describing the operational properties of the polar coordinate of the spin system. Statistical moments of the polar angle

$$\overline{\exp(in\varphi)} = \int d\Omega\, \exp(in\varphi)\,\Pr(\Omega) = \langle \hat{E}_{\mathcal{F}}^{(n)} \rangle \tag{17}$$

lead to the following operational set of polar phasors defined as

$$\hat{E}_{\mathcal{F}}^{(n)} = \int d\Omega\, e^{in\varphi}\, \hat{\mathcal{F}}(\Omega)\,. \tag{18}$$

We assume, that n is a positive integer and that $\hat{E}_{\mathcal{F}}^{(-n)} = \hat{E}^{(n)\dagger}$. Simple calculations give

$$\hat{E}_{\mathcal{F}}^{(n)} = \hat{S}_+^n \frac{\Gamma(s-\hat{S}_3+1-n/2)\Gamma(s+\hat{S}_3+1+n/2)}{\Gamma(s+\hat{S}_3+n+1)\Gamma(s-\hat{S}_3+1)}, \tag{19}$$

with $\hat{E}_{\mathcal{F}}^{(n)} = 0$ for $n > 2s$. Two first moments are given by

$$\hat{E}_{\mathcal{F}}^{(1)} = \hat{S}_+ \frac{\Gamma(s-\hat{S}_3+1/2)\Gamma(s+\hat{S}_3+3/2)}{\Gamma(s+\hat{S}_3+2)\Gamma(s-\hat{S}_3+1)}, \tag{20}$$

$$\hat{E}_{\mathcal{F}}^{(2)} = \hat{S}_+^2 \frac{s+\hat{S}_3+1}{s-\hat{S}_3}. \tag{21}$$

So far we have derived operational operators associated only with polar φ and azimuthal θ directions of the spin. In the same way, from the statistical properties of the spin propensity, it is possible to derive operational spin operators. These operators correspond to Malus measurements of unit directions with a filter defined by a spin coherent state POVM. We can parameterize the three spin coordinates by a solid angle on a unit sphere in the following way

$$\hat{S}_1 \longrightarrow \cos\varphi\sin\theta,\;\; \hat{S}_2 \longrightarrow \sin\varphi\sin\theta,\;\; \hat{S}_3 \longrightarrow -\cos\theta. \tag{22}$$

The corresponding spin operational operators may be naturally defined as follows

$$\hat{\Sigma}_i^{(n)} = \int d\Omega\, (n_i)^n\, \hat{\mathcal{F}}(\Omega)\,, \quad i = 1,2,3\,, \tag{23}$$

where $\vec{n} = (\cos\varphi\sin\theta,\, \sin\varphi\sin\theta,\, -\cos\theta)$ is a unit vector.

In further discussion we concentrate only on the first two operators from the whole operational spin algebra

$$\hat{\Sigma}_i^{(1)} = \frac{1}{1+s}\hat{S}_i, \tag{24}$$

$$\hat{\Sigma}_i^{(2)} = \frac{2}{(1+s)(3+2s)}\hat{S}_i^2 + \hat{1}\frac{1}{3+2s}. \tag{25}$$

Operational spin observables are proportional to the intrinsic spin operators, however they are modified due to the noise imposed by the measuring apparatus forming the Malus analyzer.

3. Squeezed Quantum Trigonometry

As a second example of a possible application of the presented formalism we give a theoretical description and generalization of the experiments recently performed by Noh, Fougères and Mandel [9]. The authors have used an eight-port homodyne detector (NFM apparatus) in order to measure the relative phase between two classical or quantum light fields. In such an experiment we measure the difference of the photon counts on two pairs of detectors. This quantity is either related to the sine and cosine of the phase difference of two classical electromagnetic fields (classical case) or may be used to define the set of operational operators associated with an arbitrary classical function of the phase (quantum case). Particularly, we can find quantum analogs of trigonometric functions and their powers obtaining the so called "quantum trigonometry" [11].

Below we derive such a "quantum trigonometry" for modified NFM apparatus. We will assume here, that the additional noise coming through the two free unused ports of the NFM experimental device is described by squeezed vacuum state (in the original NFM experiment this field was a vacuum state). Because of this, the resulting operational algebra will be represent by a "squeezed quantum trigonometry".

In our case the propensity $\Pr(\varphi; s, \phi) = \Pr(\varphi + 2\pi; s, \phi)$ is a periodic function of the phase, normalized to unity in the following way

$$\int \frac{d\varphi}{2\pi} \Pr(\varphi; s, \phi) = 1\,. \tag{26}$$

By s and ϕ we denoted the amplitude and the phase of the squeezed vacuum.

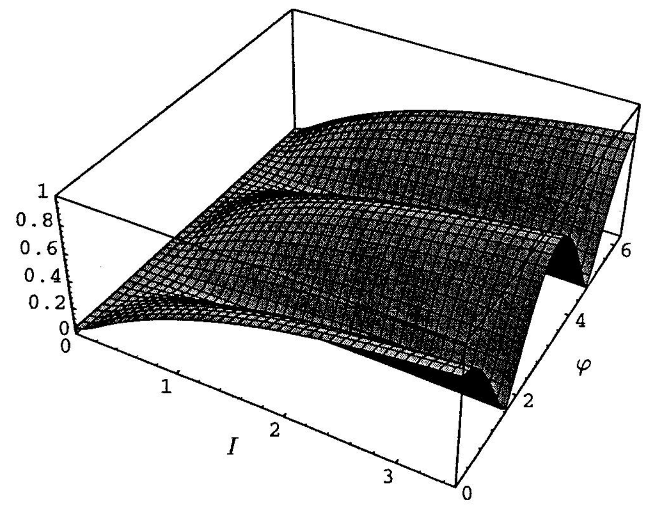

Fig. 1: Wigner function of operational operator $\hat{C}^{(2)}(s=0.5, \phi=\pi/2)$.

In accordance with the general scheme (4) the phasor operators corresponding to an operational squeezed quantum trigonometry are defined as follows

$$\overline{e^{in\varphi}} \equiv \langle \hat{E}^{(n)}(s,\phi)\rangle, \quad n=\pm 1, \pm 2, \dots . \tag{27}$$

As in [11], we assume from the beginning, that the local oscillator (reference field) is a strong coherent laser field and therefore we are allowed to neglect the noise of the input field mixed by the beam splitter with the reference signal.

We start with the formula obtained in [11]

$$\hat{E}^{(n)}(s,\phi) = \mathrm{Tr}_v \left\{ \frac{(\hat{b}+\hat{v}^\dagger)^n}{\left((\hat{b}+\hat{v}^\dagger)(\hat{b}^\dagger+\hat{v})\right)^{\frac{n}{2}}} \hat{S}(s,\phi)|0_v\rangle\langle 0_v|\hat{S}^\dagger(s,\phi) \right\}, \tag{28}$$

and change it for our purpose using a unitary operator $\hat{S}(s,\phi)$, which generates the squeezed vacuum state $|s\rangle\langle s|$ from the vacuum state $|0\rangle\langle 0|$ [12]. The bosonic creation and annihilation operators $\hat{v}^\dagger$, $\hat{v}$ represent an additional degree of freedom associated with the squeezed vacuum input at the beam splitter, by $\hat{b}$, $\hat{b}^\dagger$ we denote the creation and the annihilation operators of the signal field. Because of the fact, that the trace Tr_v in (28) is invariant under any unitary transformation, variables connected with the local oscillator can be removed and the only place in (28), where the reference field contributes is the operator $\hat{S}(s,\phi)$ with ϕ shifted by an unimportant phase that we shall ignore.

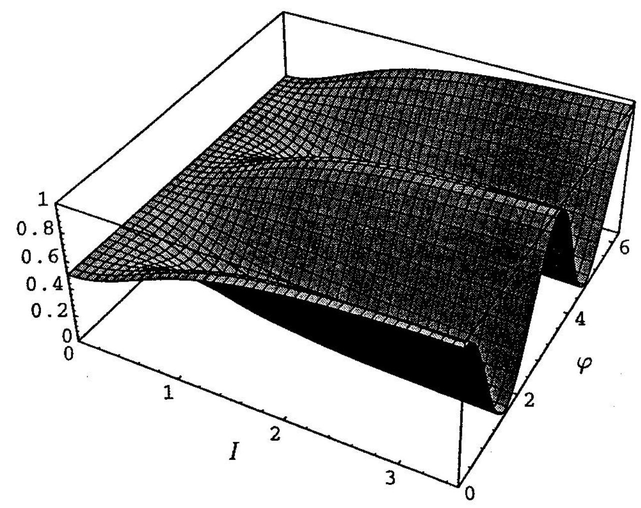

Fig. 2: Wigner function of operational operator $\hat{C}^{(2)}(s=1.5, \phi=0)$.

Using various properties of coherent and squeezed states [12], the trace in (28) can be calculated and we have

$$\hat{E}^{(n)}(s,\phi) = \int \frac{d^2w}{\pi} \frac{w^n}{(w^*w)^{\frac{n}{2}}} |w,s\rangle\langle w,s|, \tag{29}$$

where $|w,s\rangle = \hat{D}(w)\hat{S}(s,\phi)|0\rangle$ is the squeezed coherent state with amplitude w and squeezed parameters s and ϕ. According to our terminology $\hat{\mathcal{F}}(s,\phi) = |w,s\rangle\langle w,s|$ is the positive valued operator measure (POVM) associated with the described experimental scheme.

An exact formula for the phasor may be derived straight from (29). Recalling the unity decomposition for the (squeezed) coherent states we obtain

$$\hat{E}^{(n)}(s,\phi) =$$

$$\hat{S}(s,-\phi)\vdots\frac{(\hat{b}\cosh s-\hat{b}^{\dagger}e^{i\phi}\sinh s)^{n}}{\left[(\hat{b}\cosh s-\hat{b}^{\dagger}e^{i\phi}\sinh s)(\hat{b}^{\dagger}\cosh s-\hat{b}e^{-i\phi}\sinh s)\right]^{\frac{n}{2}}}\vdots\hat{S}^{\dagger}(s,-\phi). \tag{30}$$

When the squeezing parameter s tends to zero the above formula reduces to the results obtained in the reference [11].

The propensity density can be simply derived from (29). If we set $w=\sqrt{I}\exp(i\varphi)$ we obtain the propensity in the form of the following marginal integration

$$\Pr(\varphi;s,\phi)=\int_{0}^{\infty}dI\,\langle w,s|\hat{\rho}|w,s\rangle. \tag{31}$$

It's clear, that the propensity, contrary to the quantum mechanical probability density, depends on the experimental device. In fact, for each value of the squeezing parameter s we obtain a different propensity $\Pr(\varphi;s,\phi)$ and a different phasor basis, even though the probe field remains unchanged.

As it is easy to see, phasors (30) are not Hermitian operators so they cannot correspond to observable quantities. Nevertheless, using phasor basis (30) it is possible to define naturally "trigonometric operators", whose mean values can be measured in a real experiment — for $s=0$ they have been actually measured by Noh et al. [9]. For example, two first "cosine" operators are defined in the following way

$$\begin{aligned}\hat{C}^{(1)} &\equiv \frac{1}{2}(\hat{E}^{(1)}+\hat{E}^{(-1)}),\\ \hat{C}^{(2)} &\equiv \frac{1}{2}+\frac{1}{4}(\hat{E}^{(2)}+\hat{E}^{(-2)}).\end{aligned} \tag{32}$$

In a similar manner we can find moments of "sine" operators or, if it's needed, of any periodic function of the phase, provided we know its Fourier decomposition. Replacing the Fourier components $\exp(in\varphi)$ by the corresponding n-th phasors, we construct in such a way the operational operator corresponding to an arbitrary function of the phase.

In order to investigate the properties of the phasors we evaluate (numerically) the corresponding Wigner functions of these operators. Examples of such Wigner functions are presented in Fig. 1 and Fig. 2.

First we notice, that, according to the terminology introduced in [13], the phasors have a proper classical limit. If the incoming intensity of the signal field tends to infinity, the Wigner functions of the operational phasors reproduce a classical trigonometry. This limit can be seen in the Fig. 1. and in Fig. 2. Comparing both figures, we observe, that an increase of s causes a reduction of the phasors amplitude. As it might have been expected, the dependence on the squeezing parameters is gone in the classical limit.

In the limit of very small I, and with the squeezed phase ϕ equal to 0 or π we have

$$C_{W}^{(1)}(s,\phi)\overset{I\to 0}{=}\sqrt{I}\mathcal{A}_{0,\pi}(s)\cos\theta \tag{33}$$

where $\mathcal{A}_{0,\pi}$ are two amplitudes of the Wigner function, that depend only on the squeezing parameter s and $\phi=0$ or $\phi=\pi$. This result shows that the amplitude of the cosine Wigner function is literally "squeezed". For arbitrary values of ϕ the separation of the cosine Wigner function into an amplitude, and a purely angle dependent part is no longer possible, but a clear squeezing of the amplitude is also observed [14]. For $I=0$ the Wigner cosine function is zero, which is in agreement with the property that the phase of a light field in the vacuum state is randomly distributed.

Similarly we can find an asymptotic expression for the Wigner function $C_{W}^{(2)}(s,\phi)$. In the limit of small I

$$C_{W}^{(2)}(s,\phi)=\frac{1}{2}(1-c(s,\phi)), \tag{34}$$

where $c(s,\phi)$ is an I-independent function of the squeezed parameters, and its analytical expression can be derived [14]. It's easy to check, that $c(s,\phi)$ tends either to unity ($\phi=0$) or to minus unity ($\phi=\pi$). As a result, for small intensities $C_{W}^{(2)}(s,\phi)$ becomes zero ($\phi=0$) or one ($\phi=\pi$). For $\phi=\pi/2,3\pi/2$ we have $c(s,\phi)=0$ and $\lim_{I\to 0}C_{W}^{(2)}(s,\phi)=1/2$. For small I the squeezing influences the system very strongly. If the squeezed phase ϕ equals to $\pi/2$, the quadrature $\hat{C}^{(2)}$ Wigner function tends to 1/2 (Fig. 1), whereas for $\phi=0$ it takes values near zero (Fig. 2). Such a dramatic change of the cosine quadrature occurs because in the limit of small I, purely quantum effects of the squeezed vacuum are important. The squeezing allows one of the quadratures to be reduced below the vacuum level represented by a uniformly distributed random phase. The uniform distribution of the phase corresponding to the vacuum state leads to an operational Wigner function for $I=0$ equal to $\frac{1}{2}$. For a squeezed vacuum, this uniform random-phase distribution is modified [15] and a significant change of the operational quadrature is possible. In fact fluctuations below $\frac{1}{2}$ in the Wigner function exhibit the quantum nature of the squeezed vacuum.

Another interesting observation can be made if we look at the phasor's squeezed coherent states POVM decomposition (29) and recall the fact, that the Glauber P-representation of a squeezed coherent state does not exist. This property is related to the dynamical ordering of the creation and annihilation operators, induced by the measuring device. For the modified NFM apparatus, with a squeezed vacuum in the unused port, the antinormal ordering of operational phasor is impossible to achieve.

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