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EVER-EXISTING QUASIPROBABILITY FUNCTIONS BY COHERENT STATE SUPERPOSITIONS [1]

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Approximate quasiprobability functions based on discrete coherent state superpositions are introduced. It is shown that in contrast to the exact quasiprobability functions that may not exist in some region of their parameter the proposed approximate functions exist everywhere even for Gläuber P-functions.

1. Introduction

Due to the noncommuting relation between the coordinate and momentum operators, there is no unique quantum mechanical distribution function corresponding to that of classial physics where, especially in classical mechanics, phase-space methods are widely used. A whole family of quasiprobability functions can be defined according to different operator orderings [1]. For some nonclassical states these quasiprobability functions may have negative values in some region of phase space, morever in certain cases they do not exist even as a moderate distribution.

It turned out that the phase-space description is especially convenient for systems with quadratic Hamiltonians [2]. This explains why the Wigner function is so succesful in quantum optics [3]. It is also a useful tool to analyse such an inherently quantum-mechanical feature as quantum interference [4]. Recently promising attempts were carried out for using the Wigner representation in diagnostics of quantum states (the so called quantum tomography) [5].

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2. Quasiprobability functions

In phase space the statistical distribution function $W = W(x,p,t)$ determines the evolution of the statistics of the system if the initial state is specified not deterministically but in a probabilistic way [6]. This joint distribution function satisfies the Liouville equation which for a classical harmonic oscillator has the form

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + kx\frac{\partial W}{\partial p} . \tag{1}$$

The classical distribution function is a real probability function, the one-dimensional coordinate or momentum distribution can be obtained from it by integration over the conjugate variable. The expectation value of any physical quantity defined through x and p as a funtion of t is given as

$$\langle f(x,p)\rangle = \int W(x,p,t) f(x,p)\, dx dp . \tag{2}$$

For a quantum-mechanical treatment of the problem one can substitute the quantity $f(x,p)$ with a corresponding operator $f(\hat{x},\hat{p})$ on the left side of Eq. (2), however, due to the fact that the the position $\hat{x}$ and momentum $\hat{p}$ operators cannot be measured simultaneously ($[\hat{x},\hat{p}]_- \neq 0$), it is impossible to find a single quantum-mechanical probability function W. The description of the quantum-mechanical state in phase space is not unique; hence there are a family of quasiprobabilities [7], of which the Wigner [8], Glauber-Sudarshan P [9] and Husimi Q [10] functions are widely used.

The Wigner function is defined as

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\eta \exp(\alpha\eta^* - \alpha^*\eta - \frac{|\eta|^2}{2})\chi(\eta) , \tag{3}$$

where $\chi(\eta)$ is the normally ordered characteristic function

$$\chi(\eta) = \mathrm{Tr}[\hat{\rho}\exp(\eta\hat{a}^\dagger)\exp(-\eta^*\hat{a})] . \tag{4}$$

Two other quasiprobability functions, the Husimi Q function and the Glauber-Sudarshan P function can be introduced similarly

$$Q(\alpha) = \frac{1}{\pi^2}\int d^2\eta \exp(\alpha\eta^* - \alpha^*\eta - |\eta|^2)\chi(\eta) , \tag{5}$$

and

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\eta \exp(\alpha\eta^* - \alpha^*\eta)\chi(\eta) . \tag{6}$$

All these quasiprobability function can be packed into one generalized quasiprobability function [1]

$$W(\alpha,s) = \frac{1}{\pi^2}\int d^2\eta \exp(\alpha\eta^* - \alpha^*\eta - \frac{(1-s)|\eta|^2}{2})\chi(\eta) . \tag{7}$$

These functions serve as probability functions for evaluation of moments of different ordering

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}_s\rangle = \int d^2\alpha (\alpha^*)^m\alpha^n W(\alpha,s) , \tag{8}$$

where $\{(\hat{a}^\dagger)^m\hat{a}^n\}_s$ denotes normal-ordered, symmetrical-ordered or antinormal-ordered field operators for $s=1$, $s=0$ and $s=-1$ respectively.

The quasiprobability functions are not real probability distributions and they do not have to be positive or even exist at all points of the phase space. To be more specific, the Husimi function always exists and positive definite, the usual Wigner function exists, but may be negative in some regions on the α-plane, while the P function may not exist at all as a usual function or even as a moderate distribution. The fields with classical correspondence should be represented by positive, well-behaved quasiprobabilities in phase space.

3. Approximated Wigner function and quantum interference

The behaviour of the quasiprobability functions especially the existence of their negative regions are very informative on the nonclassicity of a state [11-13].

Recently Lütkenhaus and Barnett have discussed the negativity of the quasiprobabilities [12]. They introduced a quantitative measure of nonclassical behavior based on negative regions of quasiprobabilities. It has been well-known that the quasiprobabilities, especially Glauber-Sudarshan P function can be used as the measure of the nonclassicality of the given field. The nonclassical depth has been defined based on how much noise must be added to the nonclassical signal to have a positive well-defined P function [11].

Among quantum-mechanical pure states the coherent state is the only state with a definite form of the P function [12].

An even stronger aspect of quantum behaviour than the negativity of the quasiprobabilities is their nonexistence. For this, let us consider a Gaussian Wigner function, describing squeezed coherent states. The Wigner function $W_g(\alpha)$ for a general Gaussian field can be written as [14]

$$W_g(\alpha) = \frac{1}{\pi\sqrt{\tau_o^2 - 4|\mu_o|^2}}\exp\left(-\frac{\mu_o(\alpha-\alpha_o)^2 + \mu_o^*(\alpha^*-\alpha_o^*)^2 + \tau_o|\alpha-\alpha_o|^2}{\tau_o^2 - 4|\mu_o|^2}\right) . \tag{9}$$

Substituting (9) into (7) one find the generalized quasiprobability function. For the sake of simplicity let us consider a squeezed vacuum state. Its characteristic function has the form [15]

$$\chi(\eta) = \exp(-M|\eta|^2 + \frac{1}{2}S^*\eta^2 + \frac{1}{2}S\eta^{*2}) . \tag{10}$$

Here $S = \cosh r \sinh r e^{i\phi}$ and $M \geq \frac{1}{2}(\sqrt{4|S|^2+1} - 1)$, r is the squeezing parameter, the equality sign holds for pure states. The maximal and minimal uncertainties of the quadrature operator $\hat{X}_\Theta = \hat{a}e^{-i\Theta} + \hat{a}^\dagger e^{i\Theta}$ for this state are

$$\Delta X^2_{max,min} = 1 + 2M \pm 2|S| \,. \qquad (11)$$

Substituting (10) into (7) we can see that the integral in (7) exists only if

$$1 + 2M - 2|S| > s \,, \qquad (12)$$

i. e., according to (11)

$$\Delta X^2_{min} > s \,. \qquad (13)$$

As the coherent signal does not effect the relation between parameters M, S and squeezing, one can formulate a more general statement: a squeezed coherent state does not have quasiprobability functions $W(\alpha, s)$ for $s > \Delta X^2_{min}$.

The superpositions of coherent states [16,17],

$$|\, z, \phi\rangle = c_\phi(|\, z\rangle + e^{i\phi}\, |\, -z\rangle), \qquad (14)$$

referred to as Schrödinger-cat states when the constituent coherent states are macroscopically distinguishable, have attracted much interest. Although the coherent states are the most classical of all pure states of light, their simple superposition described by Eq. (14) shows remarkable nonclassical features as a consequence of the quantum interference [18-20,4]. The macroscopic superposition of coherent states [21] shows negativity in the Wigner function due to the quantum interference between the composite states. The negativity in the Wigner function can be considered to be the quantum signature of the given field.

The two most typical superposition states are the even or "male" ($\phi = 0$) and odd or "female" [22] ($\phi = \pi$) cat states. The case with small phase-space distance between the constituent states, by analogy, could be called Schrödinger-kitten states.

The characteristic function is

$$\chi(\eta) \;= |N|^2[\exp(-2|z|^2 + \eta z^* + \eta^* z) + \exp(-2|z|^2 - \eta z^* - \eta^* z) + \\ + \exp(\eta z^* - \eta^* z) + \exp(-\eta z^* + \eta^* z)]. \qquad (15)$$

The Wigner function of the male Schrödinger cat state (for real $z = x$)

$$W(\alpha) = \frac{c_+{}^2}{\pi}\left[e^{-2|\alpha - x|^2} + e^{-2|\alpha + x|^2} + 2e^{-2|\alpha|^2}\cos(4x\ \mathrm{Im}\,\alpha + \phi)\right], \qquad (16)$$

leads us to better understanding of the interference pattern. The first two terms in the Wigner function of Eq. (16) correspond to the Gaussian bells of the constituent coherent states while the third term describes an interference fringe pattern between the bells. We note that although two coherent states with strongly different arguments are almost orthogonal to each other, the maximal amplitude of the interference fringe remains two times larger than the amplitudes of the constituent coherent states, independently from the distance between them. The wavelength of the fringes decreases with the increase

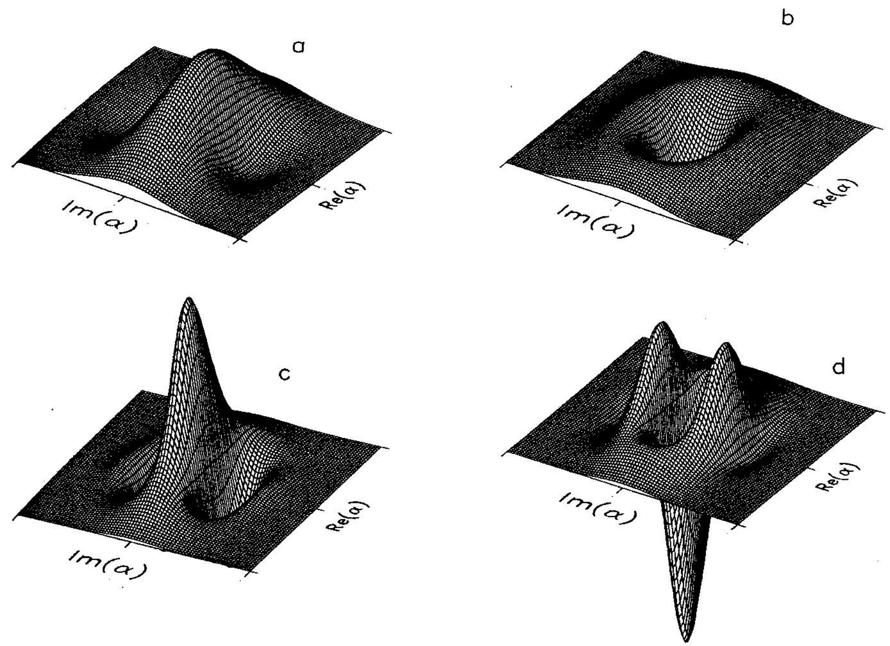

Fig. 1. Usual ($s = 0$, Fig. 1a and 1b) Wigner and generalized ($s = \frac{1}{2}$, Fig. 1c and 1d) quasiprobability functions $W(\alpha, s)$ of Schrödinger male (Fig. 1a and 1c) and female (Fig. 1b and 1d) cat states consisting of two coherent states on the real axis of the phase space. One can see that the fringe emerging from the quantum interference of the coherent states between the Gaussian bells of individual coherent states is emphasized as we change the quasiprobability function parameter s from 0 to $\frac{1}{2}$.

of the distance between the coherent states, the phase of the fringe depends on the relative phase ϕ in Eq. (14) between the composite part of the cat state.

The interference can be emphasized by chosing $s > 0$ in the generalized Wigner function

$$W(\alpha, s) \;= \frac{2|N|^2}{\pi(1-s)}\left[\exp\frac{-2(\alpha - z)(\alpha^* - z^*)}{1-s} + \exp\frac{-2(\alpha^* + z^*)(\alpha + z)}{1-s} + \right. \\ \left. + e^{-2|z|^2}\left(\exp\frac{-2(\alpha + z)(\alpha^* - z^*)}{1-s} + \exp\frac{-2(\alpha^* + z^*)(\alpha - z)}{1-s}\right)\right], \qquad (17)$$

as it is seen in Fig. 1., where two different Wigner functions with $s = 0$ (Fig. 1a and 1b) and $s = 0.5$ (Fig. 1c and 1d) are shown in the case of male (Fig. 1a and 1c) and female Schrödinger kittens with $x = 0.8$. While for the usual Wigner function the coherent states' bells merge with the interference fringe between them, for the generalized Wigner function the interference pattern can be seen clearly.

The picture becomes more complicated if we superimpose more than 2 coherent states. In this case multiple fringes can constructively or destructively interfere with

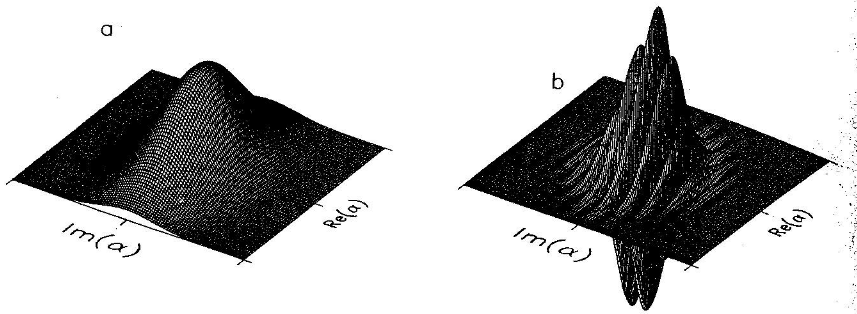

Fig. 2. Usual Wigner ($s = 0$, Fig. 2a) and approximate generalized ($s = 0.6$, Fig. 2b) quasiprobability functions $W(\alpha, s)$ of vertically squeezed vacuum state ($r = \frac{1}{3}$). Fig. 2b shows a state approximated by 12 coherent state. For these parameters r and s the exact Wigner function does not exist. In comparison with the usual (a) Wigner function the generalized quasiprobability function emphasizes the interference pattern.

each other and also with the original coherent state bells to produce different nonclassical states [18,19,23]

$$| \Psi\rangle = \sum_k f_k \mid x_k\rangle . \tag{18}$$

One can approximate any pure state with any accuracy by these discrete coherent state superpositions optimizing the number of coherent states, their positions and complex weights [23]. For example the discrete superposition of $n+1$ coherent states (a generalization of the female cat state) situated symmetrically on a circle with radius r in phase space

$$| n, r\rangle = c(r)\frac{\sqrt{n!}e^{\frac{r^2}{2}}}{(n+1)r^n}\sum_{k=0}^{n} e^{\frac{2\pi i}{n+1}k} \mid re^{\frac{2\pi i}{n+1}k}\rangle , \tag{19}$$

for small enough radius r leads to the n-photon Fock state $| n\rangle$ [24].

One can easily find the generalized approximate Wigner function

$$W_a(\alpha, s) = \frac{2}{\pi(1-s)}\sum_{l,k} f_l f_k^* \exp\left[-\frac{1}{2}|x_l|^2 - \frac{1}{2}|x_k|^2 + x_l x_k^* - \frac{2(x_l-\alpha)(x_k^*-\alpha^*)}{1-s}\right] . \tag{20}$$

A remarkable property of this appoximate quasiprobability function is that it exists for all $s < 1$, while the exact function might have a massive region of the parameter s where it ceases to exist.

Fig. 2. shows the Wigner function of a slightly squeezed vacuum state (squeezing parameter $r = \frac{1}{3}$). While for $s = 0$ (Fig. 2a) there are very few details, for $s = 0.6$ (Fig. 2b) a rather complicated quasiprobability funtion can be seen. We note that this is an appproximate Wigner function constructed with 12 equidistantly situated coherent states. For the chosen parameters ($r = \frac{1}{3}$, $s = 0.6$) the exact Wigner function does not exist. Nevertheless, using the approximate Wigner function one can find the mean value of a physical quantity with the required accuracy

$$\langle\{g(\hat{a}^\dagger, \hat{a})\}_s\rangle = \int d^2\alpha\, g(\alpha^*, \alpha)W(\alpha, s) , \tag{21}$$

where $g(\alpha^*, \alpha)$ is a whole function

$$g(\alpha^*, \alpha) = \sum_{n,m} g_{nm}(\alpha^*)^n\alpha^m . \tag{22}$$

Here as in Eq. (8) $\{\}_s$ denotes s-ordering of operators $\hat{a}^\dagger$ and $\hat{a}$. Substituting Eq. (20) into Eq. (21) one can find the required expectation values

$$\langle\{g(\hat{a}^\dagger, \hat{a})\}_s\rangle = \frac{1}{\pi}\sum_{k,k'} f_k f_{k'}^* e^{-\frac{1}{2}|x_k - x_{k'}|^2}\sum_{n,n} g_{nm}\frac{(-1)^n}{\lambda^{n+m}}\mathrm{H}_{n,m}^{(\lambda)}(x_k, -x_{k'}^*) , \quad \lambda = \frac{2}{1-s} , \tag{23}$$

where $\mathrm{H}_{n,m}^{(\lambda)}(x, y)$ are Hermite polynomials of two variables with generating function $e^{-\lambda xy}$ [25].

The approximate $W_a(\alpha, s)$ function exists everywhere but at $s = 1$. This divergence differs principally from the divergence caused by e.g. condition (13) that can be called deep divergence as it involves the nonexistence of $W(\alpha, s)$ in a whole region of s. The deep divergence was avoided by the Schrödinger kittens approximation. We shall show in the following that the remaining shallow non-existence can be removed by mixing the state with an infinitesimally small noise which does not change any physical properties of the system. For this purpose let us consider a damping process,

$$\hat{a} = \sqrt{g}\hat{a}_{in} + \sqrt{1-g}\hat{b} , \tag{24}$$

where g is damping and $\hat{b}$ the collective annihilation operator of the heatbath with mean photon number $< \hat{b}^\dagger\hat{b} >= n(T)$.

Substituting (24) into the definition of the characteristic function and that into Eq. (7) we find,

$$W_{g,n(T)}(\alpha, s) = \frac{1}{g}W^{in}(\frac{\alpha}{\sqrt{g}}, s') , \quad s' = \frac{s-(1-g)(1+2n(T))}{g} . \tag{25}$$

We can see that an infinitesimally small noise can destroy the divergence of the approximate quasiprobability function in (20).

4. Conclusions

In this paper we reviewed the properties of quantum-mechanical quasiprobability functions. We introduced approximate quasiprobability functions $W_a(\alpha, s)$ using the superposition of finite number of coherent states.

Appropriately chosen coherent state superpositions converge to a desired state very quickly due to the strong quantum interference between the constituent coherent states.

We have shown that $W_a(\alpha, s)$ exists everywhere for $s < 1$ even if the corresponding exact quasiprobability function does not. An infinitesimally small added noise can even remove this remaining divergence at $s = 1$. Using $W_a(\alpha, s)$ we derived an expression for evaluating the expectation value of $g(\hat{a}, \hat{a}^\dagger)$ operators of arbitrary ordering.

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