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DUALITY IN THE RAMSEY INTERFEROMETER[1]

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The quantum-optical Ramsey interferometer exemplifies two-way interferometers in which the beam splitter also serves the purpose of a which-way detector. It is shown that the visibility $\mathcal{V}$ of the fringes and the distinguishability $\mathcal{D}$ of the ways obey the duality relation $\mathcal{D}^2 + \mathcal{V}^2 \leq 1$ in interferometers of this kind.

1. Introduction

A recent paper [1] reports an inequality that quantifies the notion of (wave-particle) duality in the context of two-way interferometers. According to this *duality relation* — which is logically independent of uncertainty relations of the Heisenberg-Robertson kind [2] — the fringe visibility sets an absolute upper bound on the which-way information that is available in principle.

The duality relation refers to the knowability of the way. The extent to which the experimenter possesses actual knowledge of the way taken is a different matter. As a consequence of technical limitations, the experimenter may have only limited access to the available which-way information. But that is not essential because knowability counts and not human knowledge.

The derivation of the duality relation in [1] employs the simplifying assumption that different physical mechanisms are used for the various elements of the interferometer: beam splitter, phase shifter, beam merger, and which-way detector. This is the situation indicated in Fig. 1(a). The objective of the present contribution is the demonstration that the duality relation is equally valid when the same physical interaction both splits the beam and detects the way, as indicated in Fig. 1(b). Indeed, this scenario is not uncommon. For instance, Einstein's recoiling slit in a Young double-slit interferometer [3] is one realization, and so is the quantum-optical Ramsey interferometer [4]. The latter will be explored here.

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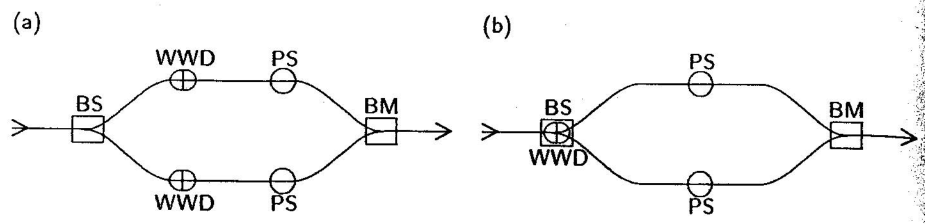

Fig. 1. Schematic two-way interferometers with which-way detectors. The beam splitter BS distributes the input among the two ways; the beam merger BM recombines the contributions and produces the output. The phase shifter PS introduces a relative phase which eventually modulates the interference pattern. In addition, there is a which-way detector WWD. In (a) another physical interaction is used for this purpose; this is the situation analyzed in [1]. In (b) the same mechanism is used both for splitting the beam and for detecting the way.

Here is a brief outline of the paper. In Sec. 2 we recall the basic equations needed to study the Ramsey interferometer. They enable us to identify, in Sec. 3, the fringe visibility $\mathcal{V}$ and the distinguishability $\mathcal{D}$ of the ways. In Sec. 4 we then convince ourselves that the duality relation [1]

$$\mathcal{D}^2 + \mathcal{V}^2 \le 1 \tag{1}$$

is obeyed. We close with a summary.

2. Ramsey interferometer

In the Ramsey interferometer of Fig. 2(a), a two-level (Rydberg) atom, prepared in its excited state, traverses two stretches of microwave radiation. The classical radiation in these Ramsey zones is assumed to be in resonance with the atomic transition. The intensity of the radiation is adjusted such that each of the zones is equivalent to a $\pi/2$ pulse. Therefore the atom would end up in its ground state if nothing else were going on. But at the central stage there is a static electric field. The differential Stark shift experienced by the atom is tantamount to a relative phase shift of ϕ between the wave function components of the two atomic states. As a consequence, the probability of ending up in the ground state equals $(1+\cos\phi)/2$. It is modulated by the phase ϕ and so exhibits Ramsey fringes (or Ramsey *beats* if one wishes to emphasize that a detuning is the origin of the ϕ dependence).

This setup is a two-way interferometer in which the ways are characterized by the natural atomic states at the central stage; see Fig. 2(b). The first microwave zone is the beam splitter, the second one is the beam merger, and the static electric field of variable strength is the phase shifter.

We describe the internal degree of freedom of the two-level atom in the usual manner with the aid of analogs of Pauli's spin operators σ_x, σ_y, and σ_z, so that $(1+\sigma_z)/2$ projects on the excited state and $(1-\sigma_z)/2$ on the ground state. Accordingly, if the

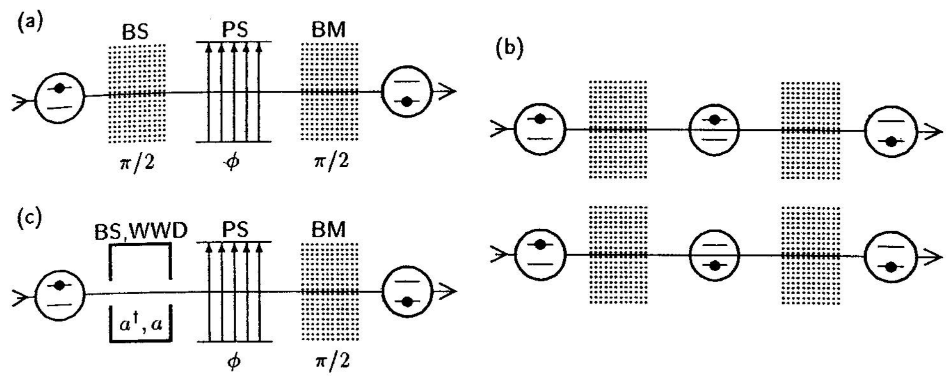

Fig. 2. Ramsey interferometer. (a) A two-level atom passes through a first microwave field that plays the role of a beam splitter BS; then it traverses a static electric field that acts as a phase shifter PS; finally a second microwave field is the beam merger BM. (b) The ways through the interferometer are characterized by the state of the atom at the central stage. In one way the atomic transition happens in the beam merger, in the other way it occurs in the beam splitter. (c) Upon replacing the classical microwave field of the beam splitter by the quantized field of a resonator, the beam splitter may also function as a which-way detector WWD. This is a realization of the abstract setup of Fig. 1(b).

atom has an initial inversion s, then its initial state is

$$\rho_{\rm at}^{(\rm i)} = \frac{1}{2}(1+s\sigma_z) \tag{2}$$

with $-1 \le s \le 1$. The action of each of the two microwave fields is given by the unitary transformation

$$\rho_{\rm at} \to \exp\Bigl(-i\frac{\pi}{4}\sigma_y\Bigr)\rho_{\rm at}\exp\Bigl(i\frac{\pi}{4}\sigma_y\Bigr)\,, \tag{3}$$

and the static electric field effects the transition

$$\rho_{\rm at} \to \exp\Bigl(-i\frac{\phi}{2}\sigma_z\Bigr)\rho_{\rm at}\exp\Bigl(i\frac{\phi}{2}\sigma_z\Bigr)\,. \tag{4}$$

The initial state (2) is thus turned into the final state

$$\rho_{\rm at}^{(\rm f)} = \frac{1}{2}\bigl[1+s(\sigma_y\sin\phi - \sigma_z\cos\phi)\bigr]\,. \tag{5}$$

The probability of finding the atom in its ground state,

$$p_\phi = {\rm tr}_{\rm at}\Bigl\{\frac{1}{2}(1-\sigma_z)\,\rho_{\rm at}^{(\rm f)}\Bigr\} = \frac{1}{2}(1+s\cos\phi)\,, \tag{6}$$

is the interference pattern of this Ramsey interferometer. Its ϕ dependence is as expected, and the fringe visibility equals the magnitude $|s|$ of the initial inversion s.

Consider now the setup of Fig. 2(c) in which the classical microwave field of the beam splitter is replaced by the quantized radiation field of a resonator. This resonator has a privileged photon mode (ladder operators $a^\dagger$, a) that is resonant with the atomic transition; all other modes are dynamically irrelevant. The two ways of Fig. 2(b) could become distinguishable after this alteration in the experiment because the number of resonator photons may undergo different changes for the two ways. Whether which-way information gets stored in the state of the resonator field or not depends on the initial photon state $\rho_{\rm ph}^{(i)}$.

The initial state of the combined atom-photon system is

$$\rho^{(i)} = \rho_{\rm at}^{(i)}\rho_{\rm ph}^{(i)} . \tag{7}$$

The interaction between the atom and the photons is well described by the familiar Jaynes-Cummings coupling (in the rotating-wave approximation). The net effect is summarized in the transition

$$\rho \to e^{-i\varphi\gamma}\rho\, e^{i\varphi\gamma} = U^\dagger \rho U ; \tag{8}$$

it replaces (3) for the beam splitter. The parameter φ is the accumulated Rabi angle and

$$\gamma = \frac{1}{2i}(\sigma_+ a - \sigma_- a^\dagger) \tag{9}$$

[with $\sigma_\pm = \sigma_x \pm i\sigma_y$ as usual] is the coupling operator with a convenient phase convention. The unitary operator of (8),

$$U = e^{i\varphi\gamma} = \frac{1+\sigma_z}{2}\cos\left(\varphi\sqrt{aa^\dagger}\right) + \frac{1-\sigma_z}{2}\cos\left(\varphi\sqrt{a^\dagger a}\right) + \frac{1}{2}\sigma_+ \frac{\sin\left(\varphi\sqrt{aa^\dagger}\right)}{\sqrt{aa^\dagger}}a - \frac{1}{2}\sigma_- a^\dagger \frac{\sin\left(\varphi\sqrt{aa^\dagger}\right)}{\sqrt{aa^\dagger}} , \tag{10}$$

exemplifies the general structure

$$U = \frac{1}{\sqrt{8}}\left[(1+\sigma_z)V_{++} + \sigma_+ V_{+-} - \sigma_- V_{-+} + (1-\sigma_z)V_{--}\right] , \tag{11}$$

where the operators $V_{++}, \ldots, V_{--}$ affect solely the degree(s) of freedom of the which-way detector, here: the photonic degree of freedom of the privileged resonator mode. The four $V_{\pm\pm}$ are restricted by the unitarity of U. These restrictions are compactly stated as the requirements that the equations

$$\begin{aligned}&(\alpha V_{++} + \beta V_{+-})^\dagger(\alpha V_{++} + \beta V_{+-}) + (\alpha V_{-+} - \beta V_{--})^\dagger(\alpha V_{-+} - \beta V_{--})\\ &= 2\alpha^*\alpha + 2\beta^*\beta \\ &= (\alpha V_{++} - \beta V_{-+})(\alpha V_{++} - \beta V_{-+})^\dagger + (\alpha V_{+-} + \beta V_{--})(\alpha V_{+-} + \beta V_{--})^\dagger\end{aligned} \tag{12}$$

hold for arbitrary complex numbers α and β.

For example, the particular choice $V_{\pm\pm} = 1$ turns U into $\exp(i\pi\sigma_y/4)$ and gets us back to the Ramsey interferometer of Fig. 2(a). Another possibility is $V_{++} = V_{-+} = U_+$, $V_{+-} = V_{--} = U_-$ with unitary operators U_+ and U_-. This characterizes the situation of Fig. 1(a) which is investigated in [1].

When supplemented by (4) and (3) for the effect of the phase shifter and the beam merger, respectively, the initial state (7) and the unitary operator U of (11) represent interferometers of the kind depicted in Fig. 1(b) in all generality. There is no need to consider initial atomic states different from (2) because all other ones are obtained by unitary transformations on the two-level degree of freedom, and these transformations amount to nothing more than a redefinition of the operators $V_{\pm\pm}$ in (11). In summary, the initial state $\rho^{(i)}$ of (7) is transformed into the final state $\rho^{(f)}$ that is of the form

$$\rho^{(f)} = \frac{1+s}{2}\rho_+^{(f)} + \frac{1-s}{2}\rho_-^{(f)} \tag{13}$$

with

$$\begin{aligned}\rho_+^{(f)} = &\frac{1+\sigma_x}{4}V_{++}^\dagger\rho_{\rm ph}^{(i)}V_{++} + \frac{1-\sigma_x}{4}V_{+-}^\dagger\rho_{\rm ph}^{(i)}V_{+-}\\ &- \frac{\sigma_z - i\sigma_y}{4}e^{-i\phi}V_{++}^\dagger\rho_{\rm ph}^{(i)}V_{+-} - \frac{\sigma_z + i\sigma_y}{4}e^{i\phi}V_{+-}^\dagger\rho_{\rm ph}^{(i)}V_{++} ,\end{aligned} \tag{14}$$

and the replacements

$$V_{++} \to -V_{-+} , \qquad V_{+-} \to V_{--} \tag{15}$$

turn $\rho_+^{(f)}$ into $\rho_-^{(f)}$.

These implications of (13) are important in the following:

a) The probability that the atom ends up in the ground state is

$$p_\phi = {\rm tr}_{\rm at}{\rm tr}_{\rm ph}\left\{\frac{1}{2}(1-\sigma_z)\rho^{(f)}\right\} = \frac{1}{2} + \frac{1}{2}{\rm Re}\left(e^{-i\phi}C\right) , \tag{16}$$

where the complex contrast factor C is given by

$$\begin{aligned}C &= {\rm tr}_{\rm ph}\left\{\frac{1+s}{2}V_{++}^\dagger\rho_{\rm ph}^{(i)}V_{+-} - \frac{1-s}{2}V_{-+}^\dagger\rho_{\rm ph}^{(i)}V_{--}\right\}\\ &= \frac{1+s}{2}\langle V_{+-}V_{++}^\dagger\rangle_{\rm ph}^{(i)} - \frac{1-s}{2}\langle V_{--}V_{-+}^\dagger\rangle_{\rm ph}^{(i)} .\end{aligned} \tag{17}$$

b) The way through the Ramsey interferometer is determined by a measurement of σ_z before the atom passes the beam merger or, equivalently, by a measurement of σ_x in the final state $\rho^{(f)}$. These two possibilities correspond to calculating the probabilities $w^{(\pm)}$ for taking the two ways by the two equivalent expressions in

$$\begin{aligned}w^{(\pm)} &= {\rm tr}_{\rm at}{\rm tr}_{\rm ph}\left\{\frac{1}{2}(1\pm\sigma_z)\exp\left(i\frac{\pi}{4}\sigma_y\right)\rho^{(f)}\exp\left(-i\frac{\pi}{4}\sigma_y\right)\right\}\\ &= {\rm tr}_{\rm at}{\rm tr}_{\rm ph}\left\{\frac{1}{2}(1\pm\sigma_x)\rho^{(f)}\right\} ,\end{aligned} \tag{18}$$

with the outcomes

$$w^{(+)} = \frac{1+s}{4}\langle V_{++}V_{++}^{\dagger}\rangle_{\rm ph}^{\rm (i)} + \frac{1-s}{4}\langle V_{-+}V_{-+}^{\dagger}\rangle_{\rm ph}^{\rm (i)},$$
$$w^{(-)} = \frac{1+s}{4}\langle V_{+-}V_{+-}^{\dagger}\rangle_{\rm ph}^{\rm (i)} + \frac{1-s}{4}\langle V_{--}V_{--}^{\dagger}\rangle_{\rm ph}^{\rm (i)}. \qquad (19)$$

c) These probabilities and also the contrast factor (17) appear in the final state of the two-level atom,

$$\rho_{\rm at}^{\rm (f)} = {\rm tr}_{\rm ph}\{\rho^{\rm (f)}\} = \frac{1}{2}\Big[1+\sigma_x\big(w^{(+)}-w^{(-)}\big) - \sigma_y{\rm Im}\big(e^{-i\phi}\mathcal{C}\big) - \sigma_z{\rm Re}\big(e^{-i\phi}\mathcal{C}\big)\Big]. \qquad (20)$$

d) If the $\sigma_z = +1$ way has been taken, then the final photon state is $\rho_{\rm ph}^{(+)}$ which is identified by

$$w^{(+)}\rho_{\rm ph}^{(+)} = {\rm tr}_{\rm at}\Big\{\frac{1+\sigma_z}{2}\rho^{\rm (f)}\Big\} = \frac{1+s}{4}V_{++}^{\dagger}\rho_{\rm ph}^{\rm (i)}V_{++} + \frac{1-s}{4}V_{-+}^{\dagger}\rho_{\rm ph}^{\rm (i)}V_{-+}, \qquad (21)$$

and likewise we get for the $\sigma_z = -1$ way

$$w^{(-)}\rho_{\rm ph}^{(-)} = {\rm tr}_{\rm at}\Big\{\frac{1-\sigma_z}{2}\rho^{\rm (f)}\Big\} = \frac{1+s}{4}V_{+-}^{\dagger}\rho_{\rm ph}^{\rm (i)}V_{+-} + \frac{1-s}{4}V_{--}^{\dagger}\rho_{\rm ph}^{\rm (i)}V_{--}. \qquad (22)$$

e) The weighted sum of $\rho_{\rm ph}^{(+)}$ and $\rho_{\rm ph}^{(-)}$,

$$\rho_{\rm ph}^{\rm (f)} = {\rm tr}_{\rm at}\{\rho^{\rm (f)}\} = w^{(+)}\rho_{\rm ph}^{(+)} + w^{(-)}\rho_{\rm ph}^{(-)}, \qquad (23)$$

is the final photon state if no determination of the way is performed (or if the result of such a measurement is deliberately ignored).

3. Fringe visibility and which-way information

The maxima and minima in the ϕ dependent pattern p_ϕ of (16) determine the fringe visibility $\mathcal{V}$ in the usual manner. This yields

$$\mathcal{V} = |\mathcal{C}| \qquad (24)$$

without further ado. The quantification of the which-way information is less familiar. As in [1] we use two numbers, the *predictability* $\mathcal{P}$ and the *distinguishability* $\mathcal{D}$ to characterize pieces of which-way knowledge of different kinds.

The predictability $\mathcal{P}$ refers to the a priori knowledge of the way. If the interferometer is operated symmetrically so that both ways are equally probable [that is: $w^{(+)} = w^{(-)} = \frac{1}{2}$], then $\mathcal{P} = 0$ because there is nothing predictable about the ways. By contrast, if the interferometer is extremely asymmetrical so that only one way is realized [that is: $w^{(+)} = 1$, $w^{(-)} = 0$ or $w^{(+)} = 0$, $w^{(-)} = 1$], then $\mathcal{P} = 1$ because the way is known with certainty beforehand. More generally we have

$$\mathcal{P} = \big|w^{(+)} - w^{(-)}\big|, \qquad (25)$$

which interpolates between these extreme situations. A given value of $\mathcal{P}$ has this significance: If one would be asked to bet on the way that the next atom will take, the winning chances are maximized by putting the money always on the more probable way. Then the actual way will be predicted correctly in the fraction $(1+\mathcal{P})/2$ of all cases.

The distinguishability $\mathcal{D}$ refers to the which-way knowledge that is stored in the final state of the which-way detector, here: in the final photon state $\rho_{\rm ph}^{\rm (f)}$ of (23). The reading of the detector amounts to measuring a suitably chosen observable A with (nondegenerate) eigenvalues A' and eigenkets $|A'\rangle$. Suppose that such a measurement has been performed and the eigenvalue A' has been found. This happens with a relative frequency that is given by

$$\langle A'|\rho_{\rm ph}^{\rm (f)}|A'\rangle = w^{(+)}\langle A'|\rho_{\rm ph}^{(+)}|A'\rangle + w^{(-)}\langle A'|\rho_{\rm ph}^{(-)}|A'\rangle, \qquad (26)$$

where the two summands correspond to the two ways. Now, again we are asked to bet on the way. The measured value of A represents additional information that potentially enables us to improve the winning chances. To this end we put the money always on the way that contributes most to the sum in (26).[3] In many repeated experiments, this strategy will yield a "*likelihood* for guessing the way right" that is given by[4]

$$\mathcal{L}_A = \sum_{A'}{\rm Max}\Big\{w^{(+)}\langle A'|\rho_{\rm ph}^{(+)}|A'\rangle, w^{(-)}\langle A'|\rho_{\rm ph}^{(-)}|A'\rangle\Big\}$$
$$= \frac{1}{2} + \frac{1}{2}\sum_{A'}\Big|w^{(+)}\langle A'|\rho_{\rm ph}^{(+)}|A'\rangle - w^{(-)}\langle A'|\rho_{\rm ph}^{(-)}|A'\rangle\Big|. \qquad (27)$$

Its calculated value can be checked if one actually determines the way — by one of the methods mentioned at (18), for instance. Such a measurement yields also the probabilities $\langle A'|\rho_{\rm ph}^{(\pm)}|A'\rangle$, and therefore the numerical value of $\mathcal{L}_A$ can be inferred from experimental data.

This value depends on the observable A that is measured. An unfortunate choice could result in $\mathcal{L}_A = \frac{1}{2}$ in which case one could just as well throw dice and would not be off worse when basing the bet on the predictability $\mathcal{P}$. Inasmuch as[5]

$$\sum_{Y'}\big|\langle Y'|X|Y'\rangle\big| \le {\rm tr}\{|X|\} \qquad (28)$$

holds for all trace-class operators X and all orthonormal sets of kets $|Y'\rangle$, the largest value of $\mathcal{L}_A$ is obtained if the (relevant) eigenkets $|A'\rangle$ of A are also eigenkets of the difference $w^{(+)}\rho_{\rm ph}^{(+)} - w^{(-)}\rho_{\rm ph}^{(-)}$. Accordingly, there is an absolute optimum for $\mathcal{L}_A$, viz.

$$\mathcal{L}_A \le \mathcal{L}_{\rm opt} = \frac{1}{2}(1+\mathcal{D}) \qquad (29)$$

[3] This betting strategy is the extension of an idea by Wootters and Zurek [5].
[4] Recall that ${\rm Max}\{x,y\} = \frac{1}{2}(x+y) + \frac{1}{2}|x-y|$ if $x, y \ge 0$.
[5] This is a variant of the Peierls inequality [6].

where

$$\mathcal{D} = \mathrm{tr}_{\mathrm{ph}}\{|w^{(+)}\rho_{\mathrm{ph}}^{(+)} - w^{(-)}\rho_{\mathrm{ph}}^{(-)}|\} \tag{30}$$

is the distinguishability. In mathematical terms, this number $\mathcal{D}$ is the distance between $w^{(+)}\rho_{\mathrm{D}}^{(+)}$ and $w^{(-)}\rho_{\mathrm{D}}^{(-)}$ in the trace-class norm; its physical significance is, however, more important: $\mathcal{D}$ is a practical, quantitative measure of the amount of which-way information that has become available. The ways cannot be distinguished at all if $\mathcal{D} = 0$, and they can be held apart completely if $\mathcal{D} = 1$.

Since the distinguishability $\mathcal{D}$ represents the which-way information acquired by an optimized reading of the final detector state, one expects that it always exceeds the predictability $\mathcal{P}$,

$$\mathcal{D} \geq \mathcal{P}\,. \tag{31}$$

This is indeed true because all trace-class operators X obey $\mathrm{tr}\{|X|\} \geq |\mathrm{tr}\{X\}|$.

As a simple illustration consider an initial photon state which contains exactly n photons. Then the operators $V_{++}, \dots, V_{--}$ identified by the comparison of (11) with (10) imply that there are no fringes, $\mathcal{V} = 0$, and the predictability and distinguishability are given by

$$\begin{aligned}\mathcal{P} &= \left|\frac{1+s}{2}\cos(2\varphi\sqrt{n+1}) - \frac{1-s}{2}\cos(2\varphi\sqrt{n})\right|,\\ \mathcal{D} &= \frac{1+s}{2}\sin^2(\varphi\sqrt{n+1}) + \frac{1-s}{2}\sin^2(\varphi\sqrt{n})\\ &\quad + \left|\frac{1+s}{2}\cos^2(\varphi\sqrt{n+1}) - \frac{1-s}{2}\cos^2(\varphi\sqrt{n})\right|.\end{aligned} \tag{32}$$

The validity of (31) can be checked explicitly.

4. The duality relation

The positivity of the statistical operator $\rho_{\mathrm{at}}^{(\mathrm{f})}$ of (20) implies immediately that the predictability and the visibility obey the inequality

$$\mathcal{P}^2 + \mathcal{V}^2 \leq 1\,. \tag{33}$$

This observation has been made by Greenberger and Yasin [7]; it is implicitly contained in the work of Wootters and Zurek [5] and also in a paper by Mandel [8].[6] The measurements by Rauch, Summhammer, and Tuppinger [9], who introduce an asymmetry into a neutron interferometer, are consistent with (33).

In view of (31) the *duality relation* (1) is more stringent than (33); examples with $\mathcal{P} = 0$ and $\mathcal{D} = 1$ can be found easily. Let us now show that (1) is obeyed by the distinguishability of (30) [with (21) and (22)] and the visibility of (24) [with (17)]. It is expedient to separate the contributions proportional to $(1+s)/2$ and $(1-s)/2$. Accordingly, we introduce

$$\text{(a) } \mathcal{C}_+ = \mathrm{tr}_{\mathrm{ph}}\{V_{++}^{\dagger}\rho_{\mathrm{ph}}^{(\mathrm{i})}V_{+-}\} \quad \text{and (b) } \mathcal{C}_- = -\mathrm{tr}_{\mathrm{ph}}\{V_{-+}^{\dagger}\rho_{\mathrm{ph}}^{(\mathrm{i})}V_{--}\} \tag{34}$$

[6] Mandel's "degree of intrinsic indistinguishability" equals $\mathcal{V}/\sqrt{1-\mathcal{P}^2}$ in the present notation.

as well as

$$\begin{aligned}\text{(a) } \mathcal{D}_+ &= \tfrac{1}{2}\mathrm{tr}_{\mathrm{ph}}\{|V_{++}^{\dagger}\rho_{\mathrm{ph}}^{(\mathrm{i})}V_{++} - V_{+-}^{\dagger}\rho_{\mathrm{ph}}^{(\mathrm{i})}V_{+-}|\}\\ \text{and (b) } \mathcal{D}_- &= \tfrac{1}{2}\mathrm{tr}_{\mathrm{ph}}\{|V_{-+}^{\dagger}\rho_{\mathrm{ph}}^{(\mathrm{i})}V_{-+} - V_{--}^{\dagger}\rho_{\mathrm{ph}}^{(\mathrm{i})}V_{--}|\}\,.\end{aligned} \tag{35}$$

Then the triangle inequality

$$\mathrm{tr}\{|X+Y|\} \leq \mathrm{tr}\{|X|\} + \mathrm{tr}\{|Y|\} \tag{36}$$

establishes

$$\mathcal{D} \leq \frac{1+s}{2}\mathcal{D}_+ + \frac{1-s}{2}\mathcal{D}_-\,, \tag{37}$$

and

$$\mathcal{C} = \frac{1+s}{2}\mathcal{C}_+ + \frac{1-s}{2}\mathcal{C}_- \tag{38}$$

holds by construction. For $s = +1$ and $s = -1$, the duality relation (1) reads

$$\mathcal{D}_+^2 + |\mathcal{C}_+|^2 \leq 1 \quad \text{and} \quad \mathcal{D}_-^2 + |\mathcal{C}_-|^2 \leq 1\,, \tag{39}$$

respectively. We shall give an explicit proof of the $s = +1$ variant; the $s = -1$ variant is then obtained by the replacements (15). And the two inequalities of (39) together, in conjunction with (37) and (38) imply the duality relation (1) right away.

For a proof of the first inequality in (39) we insert the spectral decomposition of the initial photon state,

$$\rho_{\mathrm{ph}}^{(\mathrm{i})} = \sum_k D_k |d_k\rangle\langle d_k| \tag{40}$$

[with $D_k \geq 0$, $\sum_k D_k = 1$, and $\langle d_j|d_k\rangle = \delta_{jk}$, of course], into $\mathcal{D}_+$ of (35) and arrive at

$$\mathcal{D}_+ \leq \frac{1}{2}\sum_k D_k \mathrm{tr}_{\mathrm{ph}}\{|V_{++}^{\dagger}|d_k\rangle\langle d_k|V_{++} - V_{+-}^{\dagger}|d_k\rangle\langle d_k|V_{+-}|\} \tag{41}$$

after making use of (36). For each k the trace herein can be evaluated directly[7] with the outcome

$$\mathcal{D}_+ \leq \sum_k D_k\sqrt{1-|v_k|^2} \tag{42}$$

where $v_k = \langle d_k|V_{+-}V_{++}^{\dagger}|d_k\rangle$ is a convenient abbreviation. We combine this with

$$\mathcal{C}_+ = \sum_k D_k v_k\,, \tag{43}$$

produced by inserting (40) into (34a), and get

$$\mathcal{D}_+^2 + |\mathcal{C}_+|^2 \leq \sum_{j,k} D_j D_k\left[\sqrt{1-|v_j|^2}\sqrt{1-|v_k|^2} + \frac{1}{2}v_j^* v_k + \frac{1}{2}v_k^* v_j\right]. \tag{44}$$

[7] The equality $V_{++}V_{++}^{\dagger} + V_{+-}V_{+-}^{\dagger} = 2$ is essential; it is a particular case of (12).

A look at (42) tells us that the magnitudes of the numbers v_k cannot exceed unity, $|v_k| \leq 1$; therefore the square brackets in (44) are confined to the range $0 \leq [\cdots] \leq 1$. We are thus led to

$$\mathcal{D}_+^2 + |\mathcal{C}_+|^2 \leq \sum_{j,k} D_j D_k = \left[\mathrm{tr}_{\mathrm{ph}}\{\rho_{\mathrm{ph}}^{(\mathrm{i})}\}\right]^2 = 1\,, \tag{45}$$

and this completes the proof.

We emphasize that this proof of the duality relation (1) does not rely on an uncertainty relation of the Heisenberg-Robertson kind [2], i.e., $\delta A \delta B \geq \frac{1}{2}|\langle[A,B]\rangle|$ for the spreads of two observables A and B and the expectation value of their commutator. It is equally important to realize that the inequalities (1) and (33) convey utterly different messages despite their striking similarity because the predictability $\mathcal{P}$ and the distinguishability $\mathcal{D}$ represent different pieces of which-way knowledge. Furthermore, the two inequalities concern different degrees of freedom. In (33) one meets an immediate consequence of the positivity of the final state (20) of the two-level atom. In marked contrast, (1) originates in the quantum properties of the which-way detector, here: of the resonator field.

5. Summary

The quantum-optical Ramsey interferometer of [4] exemplifies two-way interferometers in which the physical interaction of the beam splitter is also employed for the purpose of which-way detection. We have shown that the duality relation reported in [1] is equally valid for interferometers of this kind. The proof exploits the quantum properties of the which-way detector.

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