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PROPERTIES OF NULL CONGRUENCES IN SPACETIMES OF PETROV TYPE III EMBEDDED INTO E_5

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We show that if a spacetime of Petrov type III can be embedded into E_5 then their degenerate Debever-Penrose vectors must define a null geodesic congruence which is non-rotating and non-deforming. These are necessary conditions for embedding satisfied by congruences of any class-one type-III spacetime. It is not yet known, however, whether such metrics can be generated by physically admissible energy-momentum sources or not. Using our result, we show that a metric obtained by Allnut cannot be class-one.

1. Introduction.

The interest on the invariant characterization of gravitational fields independent of the choice of a coordinate system has given rise to the classification schemes of Petrov and of Churchill-Pleblański in general relativity [1–3]. The embeddings of 4-spacetimes into Euclidian spaces E_N also allow their classification according to their embedding class m —where $m = N - 4$. In this work we address the embedding problem in relation with such classification schemes. We obtain necessary conditions to be satisfied by any four-dimensional pseudo-Riemannian spacetime V_4, locally and isometrically embedded into a pseudo-Euclidian space E_5, *i.e.* a class-one V_4 [1,4,5]. If a spacetime admits such embedding then a symmetric second-fundamental-form tensor b_{ac} exist [1,3,4,6–10] which satisfies the Gauss

$$R_{ijac} = \epsilon_1 \left(b_{ia} b_{jc} - b_{ic} b_{ja} \right), \tag{1}$$

and the Codazzi equations

$$b_{ij;r} = b_{ir;j} \tag{2}$$

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where R_{ijac} is the Riemann tensor and $\epsilon_1 = \pm 1$. Additionally, we know that any type-III V_4 spacetime admits a null 3-degenerate Debever-Penrose vector. The conventions regarding tensors, Newman-Penrose (NP) quantities [2], and the Petrov and the Churchill-Plebański [3,4] classifications used in this paper are those of the well-known book of Kramer *et al.* [1]. In this communication we show that if a V_4 can be embedded into E_5 then the null Debever-Penrose vector defines a non-rotating and non-deforming geodesic congruence.

2. Necessary conditions for embedding of a Petrov type III spacetime into E_5.

From (1) and (2) we can easily calculate the Ricci tensor R_{ab} and the scalar curvature R as

$$R_{ac} = \epsilon_1 \left(b^i_a b_{ic} - b b_{ac}\right), \quad R = \epsilon_1 (b^{ij} b_{ij} - b^2) \tag{3}$$

where $b \equiv b^a_a$. The conformal tensor C_{ijkm} is

$$C_{ijkm} = R_{ijkm} - \frac{1}{2}\left(R_{ik}g_{jm} + R_{jm}g_{ik} - R_{jk}g_{im} - R_{im}g_{jk}\right) + -\frac{R}{6}\left(g_{im}g_{jk} - g_{ik}g_{jm}\right) \tag{4}$$

The conformal tensor is the key ingredient of the Petrov classification [1], C_{ijkl} may be classified in one of the six Petrov algebraic types. The Ricci tensor on the other hand may be classified in one of the 15 Churchill-Plebański types [1,3,4,16,17] —we use here Segré notation [1], so please try not to confuse the square brackets of this notation with a quoted reference. The Churchill-Plebański classification is based on the properties of a symmetrical second-rank tensor in V_4 [1–3,12,13], instead of on the properties of the fourth-rank tensor required by the former classification. In the case of 4-spacetimes we may use simultaneously the conformal tensor C_{ijkl}, the Ricci R_{ac} and the second-fundamental-form b_{ac} tensors to study relationships between both classifications and to establish, given a Petrov-type for the conformal tensor, what are the compatible types in the Churchill-Plebański classification. Such a task has been carried out by Barnes [6] in a purely algebraic investigation since no use was made of equation (2) nor of the Bianchi identities. Here we apply to Barnes results an algebraic and differential technique to obtain necessary conditions on the null Debever-Penrose congruences for any type-III V_4, if it can be embedded into E_5.

Barnes has found that for a class-one spacetime of Petrov-type III, b_{ij} has to be of type [1,3] and R_{ij} may be of either [(2,1)1], [(1,3)] or [1,3] types in the Churchill-Plebański classification. We have then 3 possibilities to analyse but, in each, we have

$$\begin{aligned} &\tilde\Phi_{00} = \tilde\Phi_{01} = \tilde\Phi_{22} = 0 \\ &\Psi_a = 0 \text{ for } a = 1 \text{ or } 2, \quad \Psi_4 = -\epsilon_1/2, \\ &\tilde\Phi_{12} = -1/(2\sqrt{2}), \quad \tilde\Phi_{02} = -2\tilde\Phi_{11}. \end{aligned} \tag{5}$$

where Ψ_a and $\tilde\Phi_{ac}$ are, respectively, NP projections of the Weyl and second-fundamental-form tensors. The values of other NP quantities depend on the Churchill-Plebański classification, we have thus three cases to analyse:

A) The case of b_{ij}:[1,3] and R_{ij}:[1(1,2)]. In this case we have

$$\begin{aligned} &R = 0, \quad \Phi_{11} = \tfrac{\epsilon_1}{2}\lambda_3^2, \quad \Phi_{02} = -2\Phi_{11}, \text{ all the others } \Phi_{ab} = 0 \\ &\Psi_3 = \tfrac{\epsilon_1}{\sqrt{2}}\lambda_3, \quad b = -2\lambda_3, \quad \tilde\Phi_{11} = \tfrac{\lambda_3}{4}, \end{aligned} \tag{6}$$

here Φ_{ac} is the NP projection of the Ricci tensor and λ_3 stands for a real eigenvalue of b_{ac}. From the Codazzi equations written in NP language and using the Bianchi identities [1,2] we may get $\kappa = 0$, $\sigma = \bar\rho$, $\epsilon = \bar\epsilon$, and $D\lambda_3 = 0$, where $\bar\rho$ is the complex conjugate of ρ and $D \equiv k^a\nabla_a$ (∇ stands for the covariant derivative). The Kundt-Thompsom theorem [1,18] together with equations (5) and (6) imply now that $\sigma = 0$ and we thence obtain the necessary conditions for embedding: In any class-one V_4 of type III with b_{ar}:[31] and R_{ac}:[(21)1], the degenerate Debever-Penrose vector specify a non-rotating, non-expanding and non-deforming null geodesic congruence. In this situation, the Codazzi equations lead also to $\epsilon = 0$, *i.e.* the mentioned geodesic congruence may be given an affine parametrization.

B) The case of b_{ij}:[1,3] and R_{ij}:[(1,3)]. In this case we have

$$\begin{aligned} &R = 0, \quad \Phi_{12} = \tfrac{\epsilon_1}{2\sqrt{2}}\lambda_3, \quad \Phi_{22} = \tfrac{\epsilon_1}{2}, \\ &b = \lambda_3, \quad \Psi_3 = \tfrac{\epsilon_1}{2\sqrt{2}}\lambda_3, \quad \tilde\Phi_{12} = -2\tilde\Phi_{11}. \end{aligned} \tag{7}$$

The Codazzi equations (2), the Bianchi identities and the Kundt-Thompson theorem imply $\kappa = \sigma = \rho = \epsilon = 0$, $\tau = \bar\tau = -2\pi$ and $D\lambda_3 = 0$. So, we conclude that the 3-degenerate Debever-Penrose null congruence is also non-rotating, non-expanding and non-deforming.

C) The case of b_{ij}:[1,3] and R_{ij}:[1,3]. In this case we have:

$$\begin{aligned} &R = -6\epsilon_1\lambda_1(\lambda_1 + \lambda_3), \quad \Phi_{11} = \tfrac{\epsilon_1}{4}\lambda_1(\lambda_1 - \lambda_3), \quad \Phi_{02} = -2\Phi_{11}, \\ &\Phi_{12} = \tfrac{\epsilon_1}{2\sqrt{2}}(\lambda_1 + \lambda_3), \quad \Phi_{22} = \tfrac{\epsilon_1}{2}, \quad \Psi_3 = \tfrac{\epsilon_1}{2\sqrt{2}}(\lambda_3 - \lambda_1), \\ &b = 3\lambda_1 + \lambda_3, \quad \tilde\Phi_{11} = \tfrac{1}{8}(\lambda_3 - \lambda_1), \end{aligned} \tag{8}$$

where λ_1 and λ_3 are the real eigenvalues of b_{ac}, $\lambda_1 \neq \lambda_3$, $\lambda_1 \neq 0$ and $\lambda_1 + \lambda_3 \neq 0$. If we repeat the analysis made in two previous cases applying Kundt-Thompson theorem, we obtain $\kappa = \sigma = \epsilon = \rho - \bar\rho = 0$, $D\lambda_1 = 0$ and $D\lambda_3 = \rho(\lambda_3 - \lambda_1)$, that is, a V_4 of type III embedded in E_5 with b_{ar}:[1,3] and R_{ar}:[1,3] has necessarily a non-deforming and non-rotating 3-degenerate Debever-Penrose null geodesic congruence. We cannot assure the absence of expansion in this case.

If we combine the conclusions for the cases A, B and C, we get our general result: *If a class-one spacetime is of type-III in the Petrov classification, then its principal 3-degenerate null congruence is geodesic with no rotation and no deformation.*

To give an example of the possible uses of our result, let us consider the spacetime metric found by Allnut [19]

$$\begin{aligned} ds^2 = \; & -(3\exp(t + 2x - 2y) + 2n^2\exp(3t + 6x))du^2 \\ & +2\exp(2t + 3x - y)du(dt + 3dx + dy) \\ & -2\exp(3t + 4x)dx^2 - \tfrac{\exp(t)}{2n^2}(\exp(2y) - 1)^{-1}dy^2, \end{aligned} \tag{9}$$

where t is a time-like coordinate, u is a null coordinate and x and y are space-like coordinates. This is a solution of Einstein equations with perfect fluid in a certain range of coordinates. It is straightforward to apply our general result to the metric (9) to conclude that it cannot be embedded into E_5. This is so because Allnut's solution has a necessarily *deforming* degenerate Debever-Penrose null congruence [19]. We are now searching for necessary conditions which metrics of Petrov types I or II should have if they could be embedded into E_5.

A further comment is in order here. Collinson [10] has shown that if solutions of Einstein equations $G_{ab} = -8\pi T_{ab}$, with T_{ab} the Maxwell energy-momentum tensor, can be embedded into E_5 then they have to be of Petrov type N. Moreover, Stephani [11–13] has shown that if T_{ab} corresponds to a perfect fluid, then the possible metrics embedded into E_5, must have Petrov types O or D. Such results have led to the question [14,15], are there any physically interesting energy-momentum sources producing type-III, class-one metrics? Not a single example is known, but there are no arguments either that may forbid their existence.

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