acta physica slovaca vol. 45 No. 5, 597 – 604 October 1995

FORCED THICKNESS VIBRATIONS OF A NON-HOMOGENEOUS PIEZOELECTRIC LAYER OF MONOCLINIC SYMMETRY WITH GENERALIZED THERMAL COUPLING

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Received 4 July 1994, accepted 11 October 1994

In the present paper, the generalized dynamical theory of thermoelasticity due to Lord and Shulman is employed to investigate the thickness vibrations of a non-homogeneous piezoelectric layer of monoclinic symmetry. The vibrations in the layer are generated owing to the application of an alternating potential and temperature difference to the faces of the layer which are assumed to be coated with infinitesimally thin electrodes. Variations of the piezoelectric potential, particle displacement and temperature with the distance along the thickness direction have been determined.

1. Introduction

Thickness vibration of a piezoelectric layer bounded by two parallel planes is a very important mode of vibration. Several researchers have attempted this problem under various kinds of physical conditions. Among these, the papers by Tiersten [1,2], Bleustein and Tiersten [3], Byrne et.al [4] etc. are worth mentioning. But none of these authors have considered the coupling of thermal field in their investigations.

The pioneering attempt in this direction was made by Krystyne Knap [5]. She studied the thickness vibration of a piezoelectric layer produced by the harmonic changes of electric potential and temperature at the bounding faces of the layer under classical thermal coupling, vide, Boley et.al. [6].

The classical thermal coupling rests upon the hypothesis that the flux of heat is proportional to the gradient of the temperature distribution. As a result of this hypothesis, which is usually referred to as the Fourier law, the temperature distribution in a body is governed by a parabolic partial differential equation which predicts that the application of a thermal disturbance in a finite region instantaneously affects all points of the body. This behaviour which implies an 'infinite speed of propagation' of thermal disturbance has been the chief reason for doubting the validity of the Fourier law and it has motivated proposals to modify the Fourier law. Such modifications were discussed by several authors like Lord and Shulman [7], Green and Lindsay [8] etc. Theories developed by these authors are called generalized thermoelasticity theories.

Application of such generalized thermoelasticity theories to piezoelectric vibration problems is quite new. Very few problems have so far been discussed. In this connection, mention may be made of the works done by Ray and Pal [9], Bassiony and Ghaleb [10], Chandrasekharaiah [11] etc.

In the present paper, an attempt has been made to investigate, in the context of the linear generalized thermoelasticity theory of Lord and Shulman [7], the thickness vibrations of a non-homogeneous piezoelectric layer of monoclinic symmetry. The vibrations are generated owing to the application of an alternating potential and temperature difference to the two faces of the layer which are assumed to be coated with infinitesimally thin electrodes. Variations of the piezoelectric potential, particle displacement and temperature with the distance along the thickness direction have been derived.

2. The fundamental equations of the problem

Newton's vibration equation, Gauss's divergence equation and the equations of state of the piezoelectric material - constitute the governing equations of the problem. The vibration equation and the divergence equation are

$$\rho \ddot{u}_i = \tau_{ij,j} \tag{2.1}$$

and

$$D_{i,i} = 0 \tag{2.2}$$

The constitutive equations of the material on which the disturbances are assumed to propagate are

$$\begin{aligned} \tau_{ij} &= C_{ijkl}S_{kl} - e_{mij}E_m - \lambda_{ij}\Theta \\ D_j &= e_{jkl}S_{kl} + \varepsilon_{ij}E_i + p_j\Theta \\ \sigma &= \lambda_{kl}S_{kl} + p_iE_i + \beta\Theta \end{aligned} \tag{2.3}$$

(vide, Mindlin [12]) where τ_{ij} are components of stress, S_{kl} are the strain components, D_i are the electric displacement components, E_i are the electric field components, σ is the entropy, u_i are the displacement components and Θ is the temperature, C_{ijkl}, e_{mij} and ε_{ij} are the elastic stiffnesses, piezoelectric constants and dielectric constants of the material. λ_{ij} and p_j are the thermoelectric and pyroelectric constants of the material, vide Mindlin [12]. β is some thermal coupling constant and ρ is the density of the medium. Here the summation convention for repeated tensor indices is employed and an index preceded by a comma denotes differentiation with respect to some space coordinate. Dot notation signifies time derivative

In addition to the equations presented above the following are also important for the problem.
The strain components

$$S_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{2.4}$$

and if ϕ is some potential function, then the electric field components are given by

$$E_i = -\phi_{,i} \tag{2.5}$$

Moreover, since we assume generalized thermal coupling as in Lord an Shulman [7], the modified form of the Fourier law can be taken in the form

$$q_i + \tau_0 \dot{q}_i = -k_{ij}\Theta_{,j} \tag{2.6}$$

The linearized energy equation is the following

$$-q_{i,i} = \Theta_0 \dot{\sigma} \tag{2.7}$$

(vide, Mindlin [12]) where q_i are the components of the heat flux vector, k_{ij} are the heat conduction coefficients, τ_0 and Θ_0 are the thermal relaxation parameter and some reference temperature.

The elastic stiffness constants C_{ijkl}, the piezoelectric constants e_{mij} appearing in the constitutive equations are with four and three indices respectively. These constants can be expressed in two-index notation, vide Mason [13]. After reducing all the constants to double index notations, we use the following matrices for the elastic, piezoelectric and dielectric constants for piezoelectric crystals of monoclinic symmetry, vide, Tiersten [2].

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \tag{2.8}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & e_{36} \end{bmatrix} \tag{2.9}$$

$$\begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \tag{2.10}$$

The matrices of all other second order tensors λ_{ij}, k_{ij} etc. have the same form as shown in equation (2.10).

As observed in the introduction, since we wish to investigate the thickness vibration of the piezoelectric layer of monoclinic symmetry, we consider the layer to be infinite in extent and bounded by two parallel planes located at $x_2 = \pm h$.
Thickness vibration corresponds to solutions which depend only on the x_2 - coordinate and so u, Θ, ϕ etc. are all functions of the x_2 - coordinate and time t and are independent

of x_1 and x_3 coordinates. Since we assume the layer to be non-homogeneous, we take the material constants in the following form

$$\{C_{22}, C_{24}, C_{44}, C_{66}, e_{26}, \varepsilon_{22}, \lambda_{22}, \lambda_{23}, k_{22}, p_2, \beta, \rho\} = \{C^0_{22}, C^0_{24}, C^0_{44}, C^0_{66}, e^0_{26}, \varepsilon^0_{22}, \lambda^0_{22}, \lambda^0_{23}, k^0_{22}, p^0_2, \beta^0, \rho^0\} e^{\alpha x_2} \tag{2.11}$$

where α is the non-homogeneity parameter.

On substituting the first equation of (2.3) in equation (2.1) and using equations (2.8), (2.9), (2.10), (2.11) and then rejecting the derivatives with respect to x_1 and x_3 coordinates, we find the following three equations

$$\begin{aligned}
\rho^0 \ddot{u}_1 &= C^0_{66}(\alpha u_{1,2} + u_{1,22}) + e^0_{26}(\alpha\phi_{,2} + \phi_{,22}) \\
\rho^0 \ddot{u}_2 &= C^0_{22}(\alpha u_{2,2} + u_{2,22}) + C^0_{24}(\alpha u_{3,2} + u_{3,22}) - \lambda^0_{22}(\alpha\Theta + \Theta_{,2}) \\
\rho^0 \ddot{u}_3 &= C^0_{24}(\alpha u_{2,2} + u_{2,22}) + C^0_{44}(\alpha u_{3,2} + u_{3,22}) - \lambda^0_{23}(\alpha\Theta + \Theta_{,2})
\end{aligned} \tag{2.12}$$

Using the second equation of (2.3) and Gauss's divergence equation (2.2) together with the corresponding coefficient matrices and (2.11) and rejecting derivatives with respect to x_1 and x_3 we find

$$e^0_{26}(\alpha u_{1,2} + u_{1,22}) - \varepsilon^0_{22}(\alpha\phi_{,2} + \phi_{,22}) + p^0_2(\alpha\Theta + \Theta_{,2}) = 0 \tag{2.13}$$

To obtain the other necessary equation we eliminate the heat flux vector q_i and the entropy σ from the last equation of (2.3) and from equations (2.6) and (2.7) and using (2.11), then rejecting as usual the derivatives with regard to x_1 and x_3 we get

$$\begin{aligned}
\frac{k^0_{22}}{\Theta_0}(\alpha\Theta_{,2} + \Theta_{,22}) &= \lambda^0_{22}(\dot{u}_{2,2} + \tau_0\ddot{u}_{2,2}) + \lambda^0_{23}(\dot{u}_{3,2} + \tau_0\ddot{u}_{3,2}) \\
&\quad - p^0_2(\phi_{,2} + \tau_0\ddot{\phi}_{,2}) + \beta^0(\dot{\Theta} + \tau_0\ddot{\Theta})
\end{aligned} \tag{2.14}$$

The three equations in (2.12) and equations (2.13) and (2.14) are the five fundamental equations of the problem.

3. Waves in the thickness direction

In the present problem, the solution will be sought in the form

$$\{u_1, u_2, u_3, \phi, \Theta\} = \{A_1, PA_1, QA_1, RA_1, TA_1\} \exp\{i(\eta x_2 - \omega t)\} \tag{3.1}$$

Now substituting the above expressions in the system of equations (2.12), (2.13) and (2.14) we find the following characteristic determinantal equation for the non-trivial values for A_1, PA_1, QA_1, RA_1 and TA_1 respectively.

$$\begin{vmatrix}
C^0_{66}(i\alpha\eta - \eta^2) + \rho^0\omega^2 & 0 & 0 & e^0_{26}(i\alpha\eta - \eta^2) & 0 \\
0 & C^0_{22}(i\alpha\eta - \eta^2) + \rho^0\omega^2 & C^0_{24}(i\alpha\eta - \eta^2) & 0 & -\lambda^0_{22}(\alpha + i\eta) \\
0 & C^0_{24}(i\alpha\eta - \eta^2) & C^0_{44}(i\alpha\eta - \eta^2) + \rho^0\omega^2 & 0 & -\lambda^0_{23}(\alpha + i\eta) \\
e^0_{26}(i\alpha\eta - \eta^2) & 0 & 0 & -\varepsilon^0_{22}(i\alpha\eta - \eta^2) & p^0_2(\alpha + i\eta) \\
0 & \lambda^0_{22}\eta\omega \times (1 - i\tau_0\omega) & \lambda^0_{23} \times \eta\omega(1 - i\tau_0\omega) & -p^0_2\eta\omega \times (1 - i\tau_0\omega) & -\{\beta^0(i\omega + \tau_0\omega^2) + (k^0_{22}/\Theta_0)(i\alpha\eta - \eta^2)\}
\end{vmatrix} = 0 \tag{3.2}$$

Substituting the values of all material constants and expanding the determinant we get a tenth degree equation of η in terms of ω.

Substituting the expressions (3.1) in equations (2.12) and (2.13) and then solving we find

$$\{P, Q, R, T\} = -\frac{1}{\Delta}\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\} \tag{3.3}$$

where Δ_i ($i = 1, 2, 3, 4$), Δ are the following determinants

$$\begin{aligned}
\Delta_1 &= |\, C_1, C_3, C_4, C_5 \,| \\
\Delta_2 &= |\, C_2, C_1, C_4, C_5 \,| \\
\Delta_3 &= |\, C_2, C_3, C_1, C_5 \,| \\
\Delta_4 &= |\, C_2, C_3, C_4, C_1 \,| \\
\Delta &= |\, C_2, C_3, C_4, C_5 \,|
\end{aligned} \tag{3.4}$$

C_i ($i = 1, 2, \ldots, 5$) are the columns of the determinant in equation (3.2) omitting the last row.

Since the determinant equation (3.2) is a tenth degree equation in η having ten complex roots η_j ($j = 1, 2, \ldots, 10$) for each value of ω, the general solution can be built up in the following form

$$\begin{aligned}
u_1 &= \sum_{j=1}^{10} A_1^{(j)} \exp\{i(\eta_j x_2 - \omega t)\} \\
u_2 &= \sum_{j=1}^{10} P_j A_1^{(j)} \exp\{i(\eta_j x_2 - \omega t)\}
\end{aligned}$$

$$
\begin{aligned}
u_3 &= \sum_{j=1}^{10} Q_j A_1^{(j)} \exp\{i(\eta_j x_2 - \omega t)\} \\
\phi &= \sum_{j=1}^{10} R_j A_1^{(j)} \exp\{i(\eta_j x_2 - \omega t)\} \\
\Theta &= \sum_{j=1}^{10} T_j A_1^{(j)} \exp\{i(\eta_j x_2 - \omega t)\}
\end{aligned} \tag{3.5}
$$

where P_j, Q_j, R_j and T_j are the values of the amplitude ratios P, Q, R and T for $\eta = \eta_j$ to be obtained from equation (3.3).

4. Boundary conditions and solution of the problem

Since in the present problem we study the vibrations produced by the harmonic changes of temperature and electric potential at the bounding surfaces of the piezoelectric layer, the boundary conditions can be taken in the following form

$$\phi = \pm\phi_0 e^{-i\omega t} \quad \text{on} \quad x_2 = \pm h \tag{4.1}$$

$$\Theta = \pm\Theta_0 e^{-i\omega t} \quad \text{on} \quad x_2 = \pm h \tag{4.2}$$

The two surfaces $x_2 = \pm h$ of the layer are assumed to be stress free. So

$$\tau_{2i} = 0 \quad (i = 1, 2, 3) \quad \text{on} \quad x_2 = \pm h \tag{4.3}$$

Using the above boundary conditions we find a set of ten equations in ten unknowns $A_1^{(j)}$. Adding and substracting these ten equations pair-wise we can write these equations in the form

$$d_{ij} A_1^{(j)} = 0, \quad (i, j = 1, 2, \ldots, 10) \tag{4.4}$$

where d_{ij}'s are the following

$$
\begin{aligned}
d_{1j} &= (C_{66}^0 + e_{26}^0 R_j)\eta_j \cos\eta_j h \\
d_{2j} &= (C_{66}^0 + e_{26}^0 R_j)\eta_j \sin\eta_j h \\
d_{3j} &= [i(C_{22}^0 P_j + C_{24}^0 Q_j)\eta_j - \lambda_{22}^0 T_j] \cos\eta_j h \\
d_{4j} &= [i(C_{22}^0 P_j + C_{24}^0 Q_j)\eta_j - \lambda_{22}^0 T_j] \sin\eta_j h \\
d_{5j} &= [i(C_{24}^0 P_j + C_{44}^0 Q_j)\eta_j - \lambda_{23}^0 T_j] \cos\eta_j h \\
d_{6j} &= [i(C_{24}^0 P_j + C_{44}^0 Q_j)\eta_j - \lambda_{23}^0 T_j] \sin\eta_j h \\
d_{7j} &= (R_j - \frac{\phi_0}{\Theta_0} T_j) \cos\eta_j h \\
d_{8j} &= (R_j - \frac{\phi_0}{\Theta_0} T_j) \sin\eta_j h \\
d_{9j} &= R_j \cos\eta_j h \\
d_{10j} &= T_j \cos\eta_j h
\end{aligned} \tag{4.5}
$$

For the non-trivial solution of the set of simultaneous equations (4.4) we find

$$| d_{ij} | = 0 \tag{4.6}$$

This is a tenth order determinant.

Now let us divide each of these equations (4.4) by $A_1^{(1)}$. Solving any nine of these equations we can find the ratios $A_1^{(j)}/A_1^{(1)}$, $j = 2, 3, \ldots, 10$. Denoting the above nine ratios by $l_{21}, l_{31}, \ldots, l_{101}$ respectively, the displacement components, piezoelectric potential and temperature can be expressed in the following form as the sum of the partial waves.

$$
\begin{aligned}
u_1 &= A_1^{(1)}[\sum_{j=1}^{10} l_{j1} \exp\{i(\eta_j x_2 - \omega t)\}] \\
u_2 &= P_1 A_1^{(1)}[\sum_{j=1}^{10} P_{j1} \exp\{i(\eta_j x_2 - \omega t)\}] \\
u_3 &= Q_1 A_1^{(1)}[\sum_{j=1}^{10} Q_{j1} \exp\{i(\eta_j x_2 - \omega t)\}] \\
\phi &= R_1 A_1^{(1)}[\sum_{j=1}^{10} R_{j1} \exp\{i(\eta_j x_2 - \omega t)\}] \\
\Theta &= T_1 A_1^{(1)}[\sum_{j=1}^{10} T_{j1} \exp\{i(\eta_j x_2 - \omega t)\}]
\end{aligned} \tag{4.7}
$$

where

$$\{P_{j1}, Q_{j1}, R_{j1}, T_{j1}\} = \left\{\frac{P_j}{P_1}, \frac{Q_j}{Q_1}, \frac{R_j}{R_1}, \frac{T_j}{T_1}\right\} l_{j1}$$

$$j = 2, 3, 4, \ldots, 10 \quad \text{and} \quad l_{11} = 1.$$

Substituting the values of the material constants in the above expressions, the variations of the displacement, piezoelectric potential and temperature with distance along the thickness direction can be evaluated.

Substituting the non-homogeneity parameter α equal to zero in the relevant equations, it can bee seen that the results obtained agree with those derived by Nandy [15] for the thickness vibration of a homogeneous generalized thermopiezoelectric layer of similar type.

5. Discussion

To summarize the above analysis, we recall that the characteristic equation (3.2) relates the wave number η and the frequency ω. The wave number and the frequency must also satisfy the determinantal equation (4.6) derived from the boundary conditions of the problem. Though, in principle, these two equations can be solved and values for the wave number η and the frequency ω can be determined which would satisfy both the determinantal equations, it is not feasible in practise owing to algebraic intricacy, vide, White and Tseng [14].

The problem can only be solved numerically by assigning a particular value for the frequency ω in the characteristic equation (3.2) and then solving it for the wave number η. The values for the wave number thus obtained together with the preassigned frequency ω are substituted into the determinantal equation (4.6) obtained from the boundary conditions to see whether the boundary conditions are satisfied. If not, a new value for the frequency ω is chosen and the process is repeated until both the equations (3.2) and (4.6) are satisfied. Once the values for frequency ω and the wave number η_j are obtained, the expressions for the displacement components, potential and temperature can be determined by substituting their values in the relevant equations.

Due to the non-availability of the material parameters for a thermopiezoelectric layer of monoclinic symmetry under generalized thermal conditions, the numerical analysis of the problem is left for sometime in future.

Acknowledgement I am grateful to Dr. A.K. Pal of Department of Mathematics, Jadavpur University for his guidance in the preparation of this paper.

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