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GENERAL SOLUTION OF CHEW–LOW EQUATIONS FOR πN SCATTERING IN TERMS OF θ–FUNCTIONS

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General solution of Chew–Low equations for πN scattering in terms of θ–functions is obtained. Previously known solutions can be derived from the new one by limiting procedures.

1. Introduction

For the first time Chew–Low equations [1] were derived for the symmetrical interaction of charged pseudoscalar mesons with fixed nucleon, which is described by source function. These equations are the subject of interest by now [1–5,7] and are the following:

$$h_i(\omega) = \frac{\lambda_i}{\omega} + \frac{1}{\pi}\int\limits_{\mu}^{\infty}\left(\frac{Imh_i(\omega')}{\omega'-\omega} + \frac{A_{ij}Imh_j(\omega')}{\omega'+\omega}\right)d\omega' \qquad (1)$$

where $\omega = \sqrt{q^2+\mu^2}$ is a meson energy with the momentum q and mass μ. Scattering amplitude is

$$h_j(\omega) = \frac{e^{i\delta_j(\omega)}sin\delta_j(\omega)}{q^3u^2(q^2)} \ , \qquad (2)$$

where $\delta_j(\omega)$ is a real scattering phase in the state with the total isotopic spin $j/2$, $u^2(q^2)$ is a Fourier transform of the source function and A_{ij} is a crossing symmetry matrix. In [4] equations of the type (1) were reformulated in terms of matrix elements of S–matrix, provided the basic properties of these elements are fundamental analytical properties of scattering amlitude. As a result equations (1)–(2) can be reduced to the following system for the functions $S_j(\omega) = e^{2i\delta_j(\omega)} = 1 + 2iq^3u^2(q^2)h_j(\omega)$:

1 $S_j(\omega)$ are analytical meromorphic functions in the complex ω-plane with the cut along real axis: $(-\infty,-1]U[1,+\infty)$.

2 $S_j^*(\omega) = S_j(\omega^*)$, i.e. $S_j(\omega)$ are real functions on the real axis.

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3 $|S_j(\omega + i0)|^2 = 1$ for all $\omega > 1$. This property is a consequense of two–particle unitarity equation on the cut $[1, +\infty)$.

4 $S_k(-\omega) = A_{kj} S_j(\omega)$. It is a crossing symmetry condition.

These properties are independent from the number of functions $S_j(\omega)$. Thus if one has a crossing symmetry matrix A_{ij} with all necessary properties he has a well defined problem to find appropriate functions $S_j(\omega)$.

Chew–Low equations for higher partial amplitudes (p,d,f,...) can also be reduced to the above system 1–4 with appropriate matrices A_{ij}. In [4] a general method of solving 1–4 with A_{ij} of arbitrary dimension was developed. In this work we will obtain by this method most general solutions of 1–4 for arbitrary scattering states.

2. s–wave solution

For the zero meson angular momentum crossing symmetry matrix is the following [4]

$$A = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 3 \end{pmatrix}. \tag{3}$$

It corresponds to isotopic spins of interacting particles 1 and 1/2. It was shown in [4] that by means of representation $S_j, j = 1, 2$ as a sum of symmetric and antisymmetric parts one can reduce a problem to the linear nonhomogeneous Riemann boundary problem. The solution of this Riemann problem is the following:

$$S(W) = \begin{pmatrix} W + \beta_0(W) - 2 \\ W + \beta_0(W) + 1 \end{pmatrix} \frac{W + \beta_0(W)}{[W + \beta_0(W)]^2 - 1} D[W(\eta)], \tag{4}$$

where $W(\omega) = \frac{1}{\pi}\arcsin\omega$, $\eta = \frac{1+iq}{\omega}$, and

$$D[W(\eta)] = \eta^{\lambda} \prod_k \frac{|\eta_k|}{\eta_k} \frac{\eta_k - \eta}{1 - \eta_k^* \eta}, \tag{5}$$

. In (5) λ is an order of a pole or a zero at the point $\omega = 0$, and η_k is a symmetrical with respect to the axis $Im\eta = 0$ and axis $Re\eta = 0$ set of zeroes $D(\omega)$ on the both sheets of it's two–sheeted Riemannian surface [4]. In (4) $\beta_0(W)$ is a meromorphic function which performs a connection between scattering amplitudes with isotopic spins 1/2 and 3/2, i.e. between S_1 and S_2. This function must satisfy the following functional equations

$$\beta_0(-W) = -\beta_0(W), \quad \beta_0(W) = \beta_0(W + 1). \tag{6b}$$

A general solution of equations (6) can be obtained in terms of theta–functions:

$$\beta_0(W) = P(\theta_1(2W)) \times$$

$$\prod_{k=1}^{N} \frac{[Q_1^{(k)}(\theta_1(2W)) + Q_2^{(k)}(\theta_2(2W)) + Q_3^{(k)}(\theta_3(W)) + Q_4^{(k)}(\theta_4(W))]}{[R_1^{(k)}(\theta_1(2W)) + R_2^{(k)}(\theta_2(2W)) + R_3^{(k)}(\theta_3(W)) + R_4^{(k)}(\theta_4(W))]} \tag{7}$$

where $P(x), Q_j^{(k)}(x), R_j^{(k)}(x), j = 1, ..., 4; k = 1, ..., N$ are polinoms with real coeffisients, and $P(-x) = -P(x), Q_1^{(k)}(-x) = Q_1^{(k)}(x), R_1^{(k)}(-x) = R_1^{(k)}(x), k = 1, ..., N$, and other polinoms are arbitrary, N is an arbitrary (but finite) integer. It is easy to see that class (7) includes particular solution of the form

$$\beta_0(W) = \frac{C}{\theta_1(2W)\theta_3(W)\theta_4(W)}, \tag{8}$$

where C is a constant. It is easy to show by the limiting prosedure that this solution turns into the solution

$$\beta_0(W) = \frac{C'}{\cos\pi W \sin\pi W}$$

which was used in [4] to explain strong dependence of scattering length from isotopic spin. Solution (8) itself also leads to the equation [4] $a_1 + 2a_3 = 0$, where $a_i, i = 1, 3$ is a scattering length. It means that solution (8) leads to a physically reasonable results.

3.Solution for arbitrary partial waves

If a meson with angular momentum l scatters on a fixed nucleon with the momentum 1/2 then crossing symmetry matrix is the following [4]

$$A_l = \frac{1}{2l+1}\begin{pmatrix} -1 & 2l+2 \\ 2l & 1 \end{pmatrix} \tag{9}$$

Let us represent $S_j, j = 1, 2$ as a sum of symmetric and antisymmetric parts

$$S(W) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} s(W) + \begin{pmatrix} -\frac{l+1}{l} \\ 1 \end{pmatrix} a(W). \tag{10}$$

Then we introduce a new unknown function $\phi(W)$. Functions $s(W), a(W)$, which are coeffisients at symmetric and antisymmetric parts of $S(W)$, depend on this new function:

$$s(w) + a(W) = \phi(W) \tag{11a}$$

$$a(W) = \frac{\phi(W)}{W + \beta_0 + 1}. \tag{11b}$$

After this one can reduce the problem of finding S_j to the following system

$$\phi(W)\phi(1 - W) = 1 \tag{12a}$$

$$\frac{\phi(w)}{l^{-1}W + \beta_0(W) + 1} = -\frac{\phi(-W)}{-l^{-1}W - \beta_0(W) + 1}. \tag{12b}$$

By taking a logarithm $g(W - \frac{1}{2}) = \log\phi(W)$ system (12) can be reduced to the functional equation

$$g(W) + g(W + 1) = \log\frac{W + \beta_0(W + \frac{1}{2}) + \frac{1}{2} + l}{W + \beta_0(W + \frac{1}{2}) + \frac{1}{2} - l}, \tag{13}$$

where $g(-W) = -g(W)$, and β_0 satisfies equation (6).

At integer l one can solve equation (13) iteratively using a representation

$$g(W) = \log \frac{W + \beta_0(W + \frac{1}{2}) + \alpha_1}{W + \beta_0(W + \frac{1}{2}) - \alpha_1} + g_\alpha(W) \quad (14)$$

where at the first iteration $\alpha_1 = \frac{1}{2} + (l-1)$ [4]. Thus we obtain

$$g_{\alpha_1}(W) + g_{\alpha_1}(W) = -\log \frac{W + \beta_0(W) + \frac{1}{2} + (l-1)}{W + \beta_0(W) + \frac{1}{2} - (l-1)} . \quad (15)$$

Then let

$$g_{\alpha_1}(W) = -\log \frac{W + \beta_0(W) + \alpha_2}{W + \beta_0(W) - \alpha_2} + g_{\alpha_2}(W) , \quad (16)$$

where $\alpha_2 = \frac{1}{2} - (l-2)$. Equations (14)–(16) define the iterative procedure. At noninteger l one can also solve (13) iteratively. In this case crossing–symmetry matrix A_{ij} does not connect with any symmetry group of transformations, with respect to which the interaction is assumed to be invariant. In fact noninteger l is a parametrization in (9).

Due to the fact that the process of solving (13) does not depend on β_0, one can put $\beta_0 = 0$. After this simplification and using (14)–(16) at arbitrary l we obtain the general solution of equation (13) in the form

$$g(W) = \log \left(\Psi(W) \frac{\Gamma\left(-\frac{(W+\frac{1}{2}+l)}{2} + 1\right) \Gamma\left(-\frac{(-W-\frac{1}{2}+l)}{2}\right)}{\Gamma\left(\frac{(W-\frac{1}{2}-l)}{2} + 1\right) \Gamma\left(-\frac{(W-\frac{1}{2}+l)}{2}\right)} \right) \quad (17)$$

which depends on the function $\Psi(W)$. This function must satisfy the following functional equations:

$$\Psi(W)\Psi(-W) = 1, \quad \Psi(W+1)\Psi(W) = -1. \quad (18b)$$

Equations (18) have the following solution in terms of theta–functions:

$$\Psi(W) = \frac{\theta_1\left(\frac{1}{2}(W - \frac{1}{2})\right)}{\theta_2\left(\frac{1}{2}(W - \frac{1}{2})\right)} . \quad (19)$$

One can rewrite (19) in terms of elliptic Jacobian functions:

$$\Psi(W) = \frac{\theta_4 \, sn\left(K(W - \frac{1}{2})\right)}{\theta_3 \, cn\left(K(W - \frac{1}{2})\right)} . \quad (20)$$

Here $\theta_i = \theta_i(0), i = 3, 4, K(k)$ is an elliptic integral of the first kind with the modulus k. In particular case, when modulus $k = 0$, we have $K = \frac{\pi}{2}, K' = \infty$, and $(\theta_4/\theta_3)^2 = k'^2 = \sqrt{1-k^2} = 1$. Then

$$\Psi(W) = \tan \frac{\pi}{2}(W - \frac{1}{2}) , \quad (21)$$

which is the result obtained in [4].

General solution (20) is a two–periodic function with the periods $4K$ and $4K + 4iK'$, $K'(k) = K(k')$. S–matrix is defined now by means of equations (9),(10).

On the basis of these results one can easily derive general solution of Chew–Low equations for arbitrary meson total momentum. In this case crossing–symmetry matrix A_{ij} is a direct product of two matrices of the type (9), provided one corresponds to the sum of meson isotopic spin 1 and nucleon isotopic spin 1/2, whereas the other corresponds to the sum of meson total momentum l and nucleon total momentum 1/2. General solution is a product of two corresponding solutions and depends on two functions of the kind (7).

4. Summary and conclusions

General solutions obtained can be used to establish the connection between parameters of class (7) solutions and scattering length experimental values as well as other experimental characteristics of low–energy πN scattering. This question is out of the scope of this paper and will be considered elsewhere.

It must be stressed that similar solutions in terms of theta–functions one can derive for crossing–symmetry matrices A_{ij} of arbitrary dimensions, which correspond to the arbitrary total momenta of a particle (meson) and a fixed particle (nucleon).

Another interesting problem is to connect quntum and corresponding classical systems following the way of [8]. It will be the subject of further investigations.

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