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THE REISSNER - NORDSTRÖM DE SITTER AND GRAVITO-DYONIC SOLUTIONS IN THE U_4 THEORY

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The Newman-Penrose formalism and the U_4 theory are used to obtain the equations of the gravitational field with spherical symmetry and torsion. For these equations, the exact solutions are given: Reissner-Nordström de Sitter and gravito-dyonic solutions.

1. Introduction

At it is well known, the Einstein's Theory of General Relativity (TGR) left unsolved some aspects concerning the study of the gravitational field, the covariant expression of the preserving laws or the interaction of the gravitational field with the fermionic field.

Aiming to extend the TGR in order to study other than the usual properties of the gravitational field, new gravitational field theories with torsion were built, theories which use a Riemann-Cartan space with independent connection and metric, spaces endowed with both curvature and torsion.

The gravitational field theories developed on Riemann-Cartan spaces think the torsion besides curvature as a fundamental measure [1], [2].

In TGR the matter appears only as a carrier for the energy-momentum, but a phenomenological description of matter, owning only energy-momentum, is insufficient for describing it's properties.

Therefore, an important problem consists in establishing the physical aspects concerning the torsion and it's microscopic origin. Thus, some theories bind torsion to the properties of the bodies which produce the gravitational field, e.g. the spin [3], [4]; other theories are motivated through grounds linked to the quantization of the gravitational field [5].

Concerning the physical meaning of the contortion three mean points of view were highlighted: a) the contortion do not propagate, thus it is an auxiliary field without physical meaning; b) the contortion propagation involves an intricate mechanism determined by some very massive particles called tordions or, c) contortion is defined by

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two massless fields with the spin two. If it is defined by these two fields, then a new interaction endowed with macroscopic effects must exist. Until now, there is no evidence for such a macroscopic reasonable source of contortion [6].

If torsion (contortion) is a real measure, experimental procedure to evidence it must exist. In [7] one can find exposed some experiments, but unfortunately, they can not be yet achieved because of the technical difficulties.

One of the used methods in order to study the gravitational field endowed with curvature and torsion is the Newman-Penrose formalism [8] which yields the exact solutions of the field equations [9], [10], [11], [12], [13].

In the present paper, by using this formalism in the frame of the U_4 theory, we derive new exact solutions for the gravitational field with spherical symmetry and torsion. For vacuum we get gravito-dyonic type solutions, and in the case of a field created by a charged body, a Reissner-Nordström-de Sitter type solution.

2. The Field Equation in U_4 Theory

In Einstein's General Relativity Theory it has adopted a space-time which retains locally, in the neighborhood of a point, the characteristics of Minkowski space-time. As such a space, Einstein chose a Riemann space, V_4, whose properties derive from the metric tensor g_{ij}; the components of this tensor are interpreted as the gravitional potentials and they determine the energy-momentum tensor.

The Riemann space V_4 of General Relativity is a curved space and of null torsion. But, in general, it is possible to choose an affine space with a metric g_{ij} independent of affine connection. Such a space has, in general, nonvanishing curvature and torsion (if the connection is nonsymmetric).

In a series of works [1], [2], [3] a generalization of Einstein's General Relativity has been proposed, by using a space-time with independent affine connection and metric, U_4, called Rieman-Cartan space.

The connection Γ^k_{ij} of U_4 space is written in the form

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + K^k_{ij} \qquad (2.1)$$

where $\left\{ {k \atop ij} \right\}$ are the Christoffel's symbols taken with respect to the metric g_{ij} and K^k_{ij} are the components of the contortion. The contortion is related by the torsion tensor S^k_{ij} by the relations

$$K^k_{ij} = -S^k_{ij} + g^{kl} g_{im} S^m_{jl} - g^{kl} g_{jm} S^m_{li} \qquad (2.2)$$

In this case, the gravitational field equations of the U_4 theory have the form [9], [10]

$$G^{ij} - \left(\nabla_k + K^l_{lk}\right)\left(K^{ijk} + g^{ij} K^{mk}_{m} - g^{ik} K^{mj}_{m}\right) = -\chi T^{ij}, \quad \chi = \text{const} \qquad (2.3)$$

where G_{ij} - is the Einstein's tensor, g_{ik} - the metric tensor and T_{ik} - the energy momentum tensor.

For the connection (2.1) and the spherical metric

$$ds^2 = -e^{2\lambda} dr^2 - r^2 (d\Theta^2 + \sin^2 \Theta d\varphi^2) + e^{2\nu} dt^2 \qquad (2.4)$$

where the function λ and ν depends only on r (static case), the following spin coefficients are chosen

$$\gamma = \frac{1}{2\sqrt{2}} \nu' e^{-\lambda} + f \quad ; \quad \varepsilon = \frac{1}{2\sqrt{2}} \nu' e^{-\lambda} + g$$
$$\rho = -\frac{1}{2\sqrt{2} r} e^{-\lambda} + p \quad ; \quad \mu = \frac{1}{2\sqrt{2} r} e^{-\lambda} + q \qquad (2.5)$$
$$\alpha = -\beta = \frac{1}{2\sqrt{2}} \frac{\cot \Theta}{r}$$

In the relations (2.5), the functions f, g, p, q are the tetradic components of the contortion which is to be determined from the field equations, and the sign " ' " represents the derivative with respect to r. The relation between K^k_{ij} and the tetradic components is given in Appendix A.

The spin coefficients (2.5) are used to write the equations of Newman-Penrose, the Bianchi's identities for the curvature and torsion [8 - 10] in order to find the elements of the curvature tensor in tetradic basis. Using the transfer equations from the tetradic basis to the coordinate one, the elements of Ricci and Einstein's tensor in coordinate basis are found.

In this condition, for the metric (2.4) and the spin coefficients (2.5), the distinct equations (2.3) in the particular case $\nu = -\lambda$ are of the form

$$e^{-2\lambda} \left[\frac{2\lambda' e^{-2\lambda}}{r} + \frac{1}{r^2} \left(1 - e^{-2\lambda}\right) - pq - \bar{p}\bar{q} \right] = -\chi T^{11}$$
$$\frac{1}{r^2} \left[\lambda'' e^{-2\lambda} - 2(\lambda')^2 e^{-2\lambda} + \frac{2\lambda' e^{-2\lambda}}{r} - \frac{1}{2} (pq - \bar{p}\bar{q} - \bar{p}q - p\bar{q}) \right] = -\chi T^{22} \qquad (2.6)$$

In the case of vacuum ($T^{\mu\nu} = 0$) the system of equations (2.6) is reduced to two equations with three unknown λ, p, q. We make the observation that, generally, the number of the field equations for spaces with torsion is smaller than the number of the unknowns. As a solution have the two equations the Schwarzschild metric which impose $p = q = 0$.

3. The Reissner - Nordström de Sitter Solution

As a source of the gravitational field with spherical symmetry and torsion we consider a body of mass m and electric charge Q. The energy - momentum tensor will have two distinct elements:

$$T^{11} = -\frac{\alpha e^{-2\lambda}}{r^4} \quad ; \quad T^{22} = \frac{\alpha}{r^6} \qquad (3.1)$$

where

$$\alpha = \frac{KQ^2}{8\pi} \quad ; \qquad K = \frac{1}{4\pi\varepsilon_0}$$

ε_0 being the vacuum electrical permitivity.

In this conditions, the equations of the gravitational field are:

$$\frac{2\lambda' e^{-2\lambda}}{r} + \frac{1}{r^2}\left(1 - e^{-2\lambda}\right) - pq - \bar{p}\bar{q} = \chi\frac{\alpha}{r^4}$$

$$\lambda'' e^{-2\lambda} - 2(\lambda')^2 e^{-2\lambda} + \frac{2\lambda' e^{-2\lambda}}{r} - \frac{1}{2}(pq - \bar{p}\bar{q} - \bar{p}q - p\bar{q}) = -\chi\frac{\alpha}{r^4} \tag{3.2}$$

If we choose the contortion equal ($p = q = iF$), the system (3.2) has the solutions (Appendix B)

$$e^{-2\lambda} = 1 - \frac{r_G}{r} + \frac{a' Q^2}{r^2} + C_4 r^2 \tag{3.3}$$

with

$$F = \left(\frac{3}{2}C_4\right)^{\frac{1}{2}} \tag{3.4}$$

From (3.4) it is noticed that, for the contortion elements, a constant is obtained. But, in the coordinate basis, the contortion elements depend on the metric. For example [14]

$$K^{234} = -\frac{4Fe^{\lambda}}{2\sqrt{2}r^2 \sin\Theta} \quad .$$

Thus, the consideration of a more general space with curvature and torsion, in the case of field generated by a body of mass m and electric charge Q, leads to the Reissner - Nordström de Sitter metric. It is noticed that the last term of the (3.3) solution appears exclusively due to torsion.

4. Exact Solution of the Gravitational Vacuum Field Equations

If the calculation of the second term of the left side of the equation (1.3) Christoffel connection is used for the calculation of the covariant derivative (for the rest contortion is rarely kept), in the same static case and with $\nu = -\lambda$, the following equations are obtained for the vacuum space [14]:

$$\frac{2\lambda' e^{-2\lambda}}{r} + \frac{1}{r^2}\left(1 - e^{-2\lambda}\right) - pq - \bar{p}\bar{q} = 0$$

$$\lambda'' e^{-2\lambda} - 2(\lambda')^2 e^{-2\lambda} + \frac{2\lambda' e^{-2\lambda}}{r} - 8fg = 0 \tag{4.1}$$

Writing $pq = \bar{p}\bar{q} = -v$, $\quad fg = u$, and the selection $u = \frac{v}{4}$, the equation system (4.1) has the solutions (Appendix C):

$$e^{-2\lambda} = 1 - \frac{r_G}{r} - \frac{a^2}{r^2} \tag{4.2}$$

From (4.1) and (4.2) results

$$u = \frac{a^2}{8r^2} \quad ; \qquad v = \frac{a^2}{2r^4} \tag{4.3}$$

One can notice that torsion is similar to a gravitational field generated by a complex charge ($a = iQ$), i.e. a gravitational monopoly. In [15], assuming the torsion as an electrodynamic potential, the equations of motion for a particle with both electrical and magnetic charge are derived. Consequently, the existence of the gravitational monopoly is suggested.

If all the functions defining the tetradic components of contortion are chosen to be real, the unique solution for vacuum, yields also as the Schwarzschild metric.

5. Conclusions

The solution indicates the fact that some cosmological models may be built without any supplementary assertions, like the insertion of a cosmological constant, but naturally, endowing the space with curvature and torsion. On such an alternative, the torsion takes over all the cosmological constants assignments. The same result was obtained in [16], but using a gauge gravitational theory. The solution shows that the space torsion can be generated by some "imaginary sources" of the gravitational field. In our opinion this result may be due to the complex mass gravito-dyon [17 - 20]. According to this idea any field source (electromagnetic or gravitational) can be written in the following complex form

$$g = g' + ig'' \quad , \qquad i = \sqrt{-1}$$

In the case of an electrical charge the real part of g is linked to the electrical charge, and the imaginary part to the magnetic monopoly. For the gravitational charge, the real part is related to the gravitational mass but the imaginary part to the gravitational monopoly. That is, the real term of the mass determines the space curvature, and the Heaviside charges (gravitational monopoly) the torsion. Therefore (4. 2) is the gravitodyonic solution of the gravitational field with spherical symmetry, curvature and torsion. Starting from the suggestion that space torsion may be generated by the Universe "dark matter" (related to the imaginary mass part term), astrophysical experiments can be settled [21 - 22]. Moreover, these experiments may indicate the very type of particles which constitute the "dark matter".

Appendix A

The components f, p, g, q are the tetradic components of the contorsion and have the form

$$K_{(221)} = 2f \qquad K_{(121)} = 2g$$
$$K_{(431)} = p \qquad K_{(341)} = \bar{p} \tag{A.1}$$
$$K_{(324)} = q \qquad K_{(423)} = \bar{q}$$

$\bar{p}, \bar{q}$ being the complex conjugates of p and q.

The relations between the tetradic components and the coordinate basis components are

$$K^{ijk} = K_{(abc)} e^{ai} e^{bj} e^{ck} \tag{A.2}$$

K_{ijk} being the contorsion components in coordinate basis and K_{abc} - in tetradic basis. e^{bj} is the contravariant tetrade coresponding to the spherical metric, having the form

$$e^{bj} = \begin{bmatrix} -e^{-\lambda} & e^{-\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{r} & -\frac{1}{r} \\ 0 & 0 & -\frac{i}{r \sin\Theta} & \frac{i}{r \sin\Theta} \\ e^{-\nu} & e^{-\nu} & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \tag{A.3}$$

Appendix B

For $p = q = iF$, the equations (3.2) become

$$\frac{2\lambda' e^{-2\lambda}}{r} + \frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + 2F^2 = \frac{a}{r^4}$$

$$\lambda'' e^{-2\lambda} - 2\lambda'^2 e^{-2\lambda} + \frac{2\lambda'}{r} e^{-2\lambda} + 2F^2 = -\frac{a}{r^4} \tag{B.1}$$

with

$$a = \alpha\chi \tag{B.2}$$

Considering $e^{-2\lambda} = y$, from (B.1) one gets

$$r^4 y'' - 2r^2 y = 4a - 2r^2 \tag{B.3}$$

With the variable change $r = e^t$, (B.3) becomes

$$\ddot{y} - \dot{y} - 2y = 4ae^{-2t} - 2 \tag{B.4}$$

where dot defines the derivate with respect to the variable t.

The solution of (B.4) has the form

$$y = 1 + ae^{-2t} + C_3 e^{-t} + C_4 e^{2t} \tag{B.5}$$

$$C_3, \quad C_4 = \text{const} \tag{B.6}$$

For $C_3 = -r_G$ (the Schwarschild radius), $a'Q^2 = a$, the solution (B.5) becomes

$$e^{-2\lambda} = 1 - \frac{r_G}{r} + \frac{a'Q^2}{r^2} + C_4 r^2 \tag{B.7}$$

Appendix C

For $pq = \bar{p}\bar{q} = -v, \quad fg = u, \quad u, \quad v$ real functions, $u = \frac{v}{4}$ and $y = e^{-2\lambda}$ the system (4.1) has the solution

$$y = C_1' + \frac{C_2'}{r} + \frac{C_3'}{r^2} \tag{C.1}$$

$$C_1', \quad C_2', \quad C_3' = \text{const} \tag{C.2}$$

Since in the case of which the space has no torsion, the equations (4.1) must have as a solution the Schwarschild metric, we choose $C_1' = 1, \quad C_2' = -r_G$. For C_3' one must choose the value

$$C_3' = -a^2 \tag{C.3}$$

Thus the general solution has the form

$$e^{-2\lambda} = 1 - \frac{r_G}{r} - \frac{a^2}{r^2} \tag{C.4}$$

If one chooses $C_3' = a^2$ the metric Reissner-Nordström is obtained, but in this case the contorsion components are zero, consequently this metric is not compatible to the torsion.

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