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THE QUANTUM BROWNIAN MOTION

Štefan Barta[1]
Department of Physics, Faculty of Electrical Engineering STU
Ilkovičova 3, 812 19 Bratislava, Slovakia

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On the basis of the exact solution of the quantum Liouville equation, the relation for the average matrix elements of the statistical operator is derived. Further it is shown that the average diagonal elements of the statistical operator obey the generalized Fokker-Planck equation for an arbitrary probability distribution of a collision force $\mathbf{f}_c(t)$.

1. Introduction

In the last ten years there has been renewed interest in the problem of describing damping of quantum system. The most common approaches to the quantum mechanical description of dissipation are based on quantum mechanical Langevin equations [1] [2] [3] or associated quantum master equation [4].

In this paper we focus our attention on a problem of the quantum Brownian motion described by the quantum Langevin equation. The thorough exposé on the history of the problem of the quantum Brownian moton is given in [5]. The Langevin equation was sucessfully used to describe the processes in quantum noise, in quantum optics and in spin relaxation theory. It can be derived from the model which consists of particle with few degrees of freedom and a "heat bath" with many degrees of freedom. Usually the heat bath consists of a set of harmonic oscillators coupled lineary to the co-ordinates $\mathbf{r}$ of the Brownian particle moving in a potential $U(\mathbf{r},t)$ [1] [2] and [6] [7] [8].

The system under study is then governed by the Hamiltonian

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + U(\mathbf{r},t) + \sum_{n=1}^{n=N} \{ \frac{\hat{\mathbf{p}}_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 (\mathbf{x}_n - \frac{c_n}{m_n \omega_n^2} \mathbf{r})^2 \}, \qquad (1)$$

where $\mathbf{p}, \mathbf{r}$ are generalized co-ordinates and momentums of the Brownian particle and $\{\mathbf{p}_n, \mathbf{x}_n\}$ are generalized co-cordinates and momentums of harmonic oscillators.

The model of Hamiltonian is quite good in quantum noise theory, in quantum optics or in solid state theory since the relevant heat bath is the electromagnetic field or

[1] E-mail address: bartas@elf.stuba.sk

vibration of atoms, which are exactly equivalent to such an assembly of oscillators. Until now this model of Hamiltonian is not exactly solved and,therefore, one has to use some kind of approximations. Before we formulate the approximation under which the problem of quantum Brownian motion of particle will be solved, we will derive the quantum Langevin equation.

Proceeding analogously as in [9] we can write the following Heisenberg equations

$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{i}{\hbar}[\hat{H},\mathbf{r}] = \frac{\hat{\mathbf{p}}(t)}{m}, \tag{2}$$

$$\frac{d\hat{\mathbf{p}}(t)}{dt} = \frac{i}{\hbar}[\hat{H},\hat{\mathbf{p}}] = \hat{\mathbf{f}}_d(\hat{\mathbf{r}}(t),t) - m\Gamma^2\hat{\mathbf{r}}(t) + \sum_{n=1}^{n=N} c_n\hat{\mathbf{x}}_n(t), \tag{3}$$

$$\frac{d\hat{\mathbf{x}}_n}{dt} = \frac{i}{\hbar}[\hat{H},\mathbf{x}_n] = \frac{\hat{\mathbf{p}}_n(t)}{m_n}, \tag{4}$$

$$\frac{d\hat{\mathbf{p}}_n(t)}{dt} = \frac{i}{\hbar}[\hat{H},\mathbf{p}_n] = -m_n\omega_n^2\hat{\mathbf{x}}_n(t) + c_n\hat{\mathbf{r}}(t), \tag{5}$$

where $\hat{\mathbf{f}}_d(\hat{\mathbf{r}}(t),t) = -\nabla U(\mathbf{r}(t),t)$, $m\Gamma^2 = \sum_{n=1}^{n=N}\frac{c_n^2}{m_n\omega_n^2}$.

From the equations (4) and (5) follows the equation

$$\frac{d^2\hat{\mathbf{x}}_n(t)}{dt^2} = -\omega_n^2\hat{\mathbf{x}}_n(t) + \frac{c_n}{m_n}\hat{\mathbf{r}}(t). \tag{6}$$

The solution of equation (6) is the following

$$\hat{\mathbf{x}}_n(t) = \frac{\hat{\mathbf{p}}_n(0)}{m_n\omega_n}\sin\omega_n t + \hat{\mathbf{x}}_n(0)\cos\omega_n t + \frac{c_n}{m_n}\int_0^t \frac{\sin\omega_n(t-\tau)}{\omega_n}\hat{\mathbf{r}}(\tau)d\tau. \tag{7}$$

From equations (2) and (3) follows the equation

$$\frac{d^2\hat{\mathbf{r}}(t)}{dt^2} = \frac{1}{m}\hat{\mathbf{f}}_d(\hat{\mathbf{r}}(t),t) - \Gamma^2\hat{\mathbf{r}}(t) + \frac{1}{m}\sum_{n=1}^{n=N} c_n\hat{\mathbf{x}}_n(t). \tag{8}$$

Introducing relation (7) into equation (8) we obtain the equation

$$\frac{d^2\hat{\mathbf{r}}(t)}{dt^2} = -\frac{1}{m}\int_0^t f(t-\tau)\frac{d\hat{\mathbf{r}}(\tau)}{d\tau}d\tau + \frac{1}{m}[\hat{\mathbf{f}}_d(\hat{\mathbf{r}}(t),t) + \hat{\mathbf{f}}_c(t)] - \frac{1}{m}f(t)\hat{\mathbf{r}}(0), \tag{9}$$

where

$$\hat{\mathbf{f}}_c(t) = \sum_{n=1}^{n=N}\{\frac{c_n}{m_n\omega_n}\hat{\mathbf{p}}_n(0)\sin\omega_n t + c_n\hat{\mathbf{x}}_n(0)\cos\omega_n t\}, \tag{10}$$

$$f(t) = \sum_{n=1}^{n=N}\frac{c_n^2}{m_n\omega_n^2}\cos\omega_n t. \tag{11}$$

The function $f(t)$ can be seen to have the form of a memory function, since it makes the equation of motion at time t which depends on the values of $\frac{d\hat{\mathbf{r}}(t)}{dt}$ for previous time and, therefore, equation (9) represents the quantum Langevin equation with the memory.

It can be seen from equation (9) that the operator $\hat{\mathbf{f}}_c(t)$ in fact depends only on the initial bath operators and from this fact it follows that the operator $\hat{\mathbf{f}}_c(t)$ is a random operator , where its statistics is defined by the density operator at $t = 0$. Further, it can be shown that the operators $\hat{\mathbf{f}}_c(t)$ and $\hat{\mathbf{f}}_c(t')$ fulfil the following commutation relation:

$$\hat{\mathbf{f}}_c(t)\hat{\mathbf{f}}_c(t') - \hat{\mathbf{f}}_c(t')\hat{\mathbf{f}}_c(t) = i\hbar\frac{df(t-t')}{dt}\mathcal{I}, \tag{12}$$

where $\mathcal{I}$ is a unit tensor.

Due to the fact that the operators $\hat{\mathbf{f}}_c(t)$ and $\hat{\mathbf{f}}_c(t')$ are non-commuting operators, the problem of the quantum Brownian motion is not exactly solved until now. But on the other hand, we can calculate all symmetrized moments or all cumulants of the random operator $\hat{\mathbf{f}}_c(t)$ [10]. We will give, as an example, the calculation of the first and second cumulants in a special case when the initial density operator of the heat bath is given by the relation

$$\hat{\rho}(t=0) = Z^{-1}e^{-\frac{1}{k_BT}\sum_{n=1}^{n=N}\hat{H}_n}, \tag{13}$$

where $\hat{H}_n = \frac{1}{2}\frac{\hat{\mathbf{p}}_n^2}{m_n} + \frac{1}{2}m_n\omega_n^2\mathbf{x}_n^2$, and

$$Z = Spe^{-\frac{1}{k_BT}\sum_{n=1}^{n=N}\hat{H}_n}.$$

The first and second cumulants are expressed by the relations:

$$<\hat{\mathbf{f}}_c(t)>_c = Sp\hat{\mathbf{f}}_c(t)\hat{\rho}(t=0) = 0, \tag{14}$$

$$\begin{aligned} <\hat{\mathbf{f}}_c(t),\hat{\mathbf{f}}_c(t')>_c &= K(t-t')\mathcal{I} = \\ &= Sp\frac{1}{2}[\hat{\mathbf{f}}_c(t)\hat{\mathbf{f}}_c(t') + \hat{\mathbf{f}}_c(t')\hat{\mathbf{f}}_c(t)]\hat{\rho}(t=0) = \\ &= \frac{1}{2}\sum_{n=1}^{n=N}\frac{c_n^2}{m_n\omega_n}\hbar\coth\frac{\hbar\omega_n}{2k_BT}\cos\omega_n(t-t')\mathcal{I}, \end{aligned} \tag{15}$$

where $K(t-t')$ is a stationary correlation function.

If we let $\hbar \to 0$, then according to relation (12) $\hat{\mathbf{f}}_c(t)$ commutes with $\hat{\mathbf{f}}_c(t')$, and using the limit

$$\lim_{\hbar\to 0}\hbar\coth\frac{\hbar\omega_n}{2k_BT} = \frac{2k_BT}{\omega_n}$$

in (15) we obtain

$$K(t-t') = k_BT\sum_{n=1}^{n=N}\frac{c_n^2}{m_n\omega_n}\cos\omega_n(t-t') = k_BTf(t-t'). \tag{16}$$

In some cases the assembly of harmonic oscillators have the following special properties:

There is a smooth dense spectrum of oscillators frequences, i.e. the frequence spectrum of harmonic oscillators is quasi-continnous.

The coupling constants c_n of the system to the bath operators and m_n are the smooth functions of oscillator frequences.

If the above-mentioned conditions are fulfilled then we can write

$$\sum_n \frac{c_n^2}{m_n\omega_n^2} f(\omega_n) \Rightarrow \int_0^\infty g(\omega)\frac{4m^2D\gamma^2(\omega)}{m(\omega)\omega^2} f(\omega)d\omega,$$

where $g(\omega)$ is a spectral density of oscillators, $c_n = 2m\sqrt{D}\gamma_n$.

Introducing the above-mentioned transition from the summation to integral-form into relation (15) we obtain

$$< \hat{\mathbf{f}}_c(t)\hat{\mathbf{f}}_c(t') >_c \; = \; 2m^2D\int_0^\infty g(\omega)\frac{\gamma^2(\omega)}{m(\omega)\omega}\hbar\coth\frac{\hbar\omega}{2k_BT}\cos\omega(t-t')d\omega\mathcal{I} = \\ = 2m^2DW(t-t')\mathcal{I}. \quad (17)$$

In the further text we will consider the situation when

$$g(\omega)\frac{\gamma^2(\omega)}{m(\omega)\omega^2} = \frac{1}{\pi k_BT}.$$

In this case relation (11) can be written in the form

$$f(t) = \frac{4m^2D}{\pi k_BT}\int_0^\infty \cos\omega t d\omega = \frac{2m^2D}{k_BT}\delta(t), \quad (18)$$

where $\delta(t)$ is a Dirac delta function.

Introducing relation (18) into equation (9) we obtain the following equation

$$\frac{d^2\hat{\mathbf{r}}(t)}{dt^2} = -\beta\frac{d\hat{\mathbf{r}}(t)}{dt} + \frac{1}{m}\hat{\mathbf{f}}(\hat{\mathbf{r}}(t),t), \quad (19)$$

where $\hat{\mathbf{f}}(\hat{\mathbf{r}}(t),t) = \hat{\mathbf{f}}_d(\hat{\mathbf{r}}(t),t) + \hat{\mathbf{f}}_c(t)$ and $\beta = \frac{mD}{k_BT}$.

2. Solution of the quantum Liouville equation

For the formulation of the quantum Liouville equation it is necessary to know the Hamiltonian of the Brownian particle. The Hamiltonian has to be constructed in such way that the Heisenberg equation is the same as equation (19). This requirement can be fulfilled in the following way: When we treated the Brownian particle as the classical particle then equation (19) has the form

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\beta\mathbf{v}(t) + \frac{1}{m}[\mathbf{f}_d(\mathbf{r}(t),t) + \hat{\mathbf{f}}_c(t)] \quad (20)$$

It is possible to show [11] that the Langrangian function is expressed by the relation

$$\mathcal{L} = e^{\beta t}[\frac{m}{2}\mathbf{v}^2 - U(\mathbf{r},t) + \mathbf{r}.\hat{\mathbf{f}}_c(t)]. \quad (21)$$

On the basis of relation (23), the Hamiltonian function can be written in the form

$$\mathcal{H} = \mathbf{p}.\mathbf{v} - \mathcal{L} = e^{-\beta t}\frac{\mathbf{p}^2}{2m} + e^{\beta t}(U(\mathbf{r},t) - \mathbf{r}.\hat{\mathbf{f}}_c(t)), \quad (22)$$

where $\mathbf{p} = \nabla_{\mathbf{v}}\mathcal{L} = e^{\beta t}m\mathbf{v}$ is the generalized momentum.
From relation (22) immediately follows the Hamiltonian, which has the form

$$\hat{H} = -e^{-\beta t}\frac{\hbar^2}{2m}\Delta + e^{\beta t}(U(\mathbf{r},t) - \mathbf{r}.\hat{\mathbf{f}}_c(t)). \quad (23)$$

It can be shown that using the Hamiltonian (23) the Heisenberg equation is the same as equation (19).

In the general case, the external force $\mathbf{f}_d$ may depend on $\mathbf{r}$ and t. In this paper we shall consider that the external force will be a function only of the time variable.

Further, we will use some approximation which will lie in the following fact. The Brownian particle will be treated quantum-mechanically but the random force $\hat{\mathbf{f}}_c(t)$ will be treated classicaly ($\hbar \to 0$). In the framework of this approximation instead of operator $\hat{\mathbf{f}}_c(t)$ we will use the classical quantity $\mathbf{f}_c(t)$. For improving this approximation we will substitute the classical cumulants in the obtained results by the quantum cumulants. Because the force $\mathbf{f_c}(t)$ is a random vector we must, at first, define the force $\mathbf{f_c}(t)$ by the probability distribution and then give the statistical description of the dynamics of the particle. Usually it is assumed that the probability distribution of the $\mathbf{f_c}(t)$ is Gaussian. In this paper we shall consider in the certain cases that the probability distribution is also Gaussian with the first two statistical moments expressed by relations (14) and (17).

As the rule, the δ- correlation function is considered more often. This type of the correlation function can be obtained by introducing relation (18) into relation (16).

The generalized Fokker-Planck equation will be derived in the case of the arbitrary probability distribution of the $\mathbf{f_c}(t)$.

The statistical description of the Brownian motion will be done through the average matrix element of the statistical operator $\hat{\rho}(t)$. The average matrix element of the $\hat{\rho}(t)$ is defined by the following relation:

$$<< \mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_a >> \stackrel{\text{def}}{=} \int_{t\le\tau\le t_0} < \mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_a > W[\mathbf{f}^t_{c_{t_0}}(\tau)]\Pi d\mathbf{f}_c(\tau), \quad (24)$$

where $W[\mathbf{f}^t_{c_{t_0}}]$ is the functional probability distribution of the collision force $\mathbf{f}_c(t)$ over the time interval (t,t_0).

In the subsequent sections we shall evaluate exactly the average matrix elements of the $\hat{\rho}(t)$. Then we shall derive the generalized Fokker-Planck equation for the average diagonal matrix element of the $\hat{\rho}(t)$.

The statistical operator $\hat{\rho}(t)$ obeys the quantum Liouville equation:

$$i\hbar\frac{\partial\hat{\rho}(t)}{\partial t} = [\hat{H}, \hat{\rho}(t)]. \tag{25}$$

The solution of equation (25) can be written in the form

$$\hat{\rho}(t) = \hat{T}(t,t_0)\hat{\rho}(t=0)\hat{T}^{\dagger}(t,t_0), \tag{26}$$

where $\hat{T}(t,t_0)$ is the time-evolution operator, which obeys the equation

$$i\hbar\frac{\partial\hat{T}(t,t_0)}{\partial t} = \hat{H}\hat{T}(t,t_0), \tag{27}$$

$\hat{T}^{\dagger}(t,t_0)$ is the Hermitian adjoint operator of the $\hat{T}(t,t_0)$.

On the basis of relation (26), matrix element of the $\hat{\rho}(t)$ is expressed through the Feynman's path integrals [12] in the following way:

$$\begin{aligned} <\mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_a> &= \int\int d\mathbf{y}d\mathbf{z} <\mathbf{r}_b|\hat{T}(t,0)|\mathbf{y}><\mathbf{y}|\hat{\rho}(t=0)|\mathbf{z}><\mathbf{z}|\hat{T}^{\dagger}(t,0)|\mathbf{r}_a>= \\ &= \int\int d\mathbf{y}d\mathbf{z}G(\mathbf{r}_b,t;\mathbf{y},0) <\mathbf{y}|\hat{\rho}(t=0)|\mathbf{z}> G^*(\mathbf{r}_a,t;\mathbf{z},0), \end{aligned} \tag{28}$$

where

$$G(\mathbf{r}_b,t;\mathbf{y},0) = \int_{\mathbf{y}}^{\mathbf{r}_b} e^{\frac{i}{\hbar}\int_0^t e^{\beta\tau}[\frac{m}{2}\dot{\mathbf{r}}^2(\tau)+\mathbf{f}(\tau).\mathbf{r}(\tau)]d\tau}\mathcal{D}[\mathbf{r}(\tau)] \tag{29}$$

and

$$G^*(\mathbf{r}_a,t;\mathbf{y},0) = \int_{\mathbf{z}}^{\mathbf{r}_a} e^{-\frac{i}{\hbar}\int_0^t e^{\beta\tau}[\frac{m}{2}\dot{\mathbf{r}}^2(\tau)+\mathbf{f}(\tau).\mathbf{r}(\tau)]d\tau}\mathcal{D}[\mathbf{r}(\tau)]. \tag{30}$$

The evaluation of integrals (29) and (30) is performed in the following way: In the path integral (29) we choose the transformation

$$\mathbf{r}(\tau) = \mathbf{v}(\tau) + \mathbf{u}(\tau) \tag{31}$$

where $\mathbf{u}(\tau)$ is given by the solution of the equation

$$\ddot{\mathbf{u}}(\tau) = -\beta\dot{\mathbf{u}}(\tau) + \frac{\mathbf{f}(\tau)}{m} \tag{32}$$

for the initial conditions

$$\mathbf{u}(0) = \dot{\mathbf{u}}(0) = 0. \tag{33}$$

The solution of equation (32) by considering (33) is expressed by the relation

$$\mathbf{u}(t) = \int_0^t e^{-\beta\tau_1}\int_0^{\tau_1} e^{\beta\tau_2}\frac{\mathbf{f}(\tau_2)}{m}d\tau_1 d\tau_2. \tag{34}$$

After substitution relation (31) into relation (29) and considering equation (32) and relation (34) the following relation is obtained:

$$\begin{aligned} G(\mathbf{r}_b,t;\mathbf{y},0) = {} & e^{\frac{i}{\hbar}e^{\beta t}m\dot{\mathbf{u}}(t).\mathbf{r}_b - \frac{i}{2\hbar}e^{\beta t}m\dot{\mathbf{u}}(t).\mathbf{u}(t)} \\ & e^{\frac{i}{2\hbar}\int_0^t e^{\beta\tau}\mathbf{f}(\tau).\mathbf{u}(\tau)d\tau}G_0(\mathbf{r}_b - \mathbf{u}(t),t;\mathbf{y},0) \end{aligned} \tag{35}$$

where

$$G_0(\mathbf{r}_1,t;\mathbf{r}_2,0) = \int_{\mathbf{r}_2}^{\mathbf{r}_1} e^{\frac{i}{\hbar}\int_0^t e^{\beta t}\frac{m}{2}\dot{\mathbf{v}}^2(\tau)d\tau}\mathcal{D}[\mathbf{r}(\tau)]$$

is the propagator which obeys the equation

$$i\hbar\frac{\partial G_0(\mathbf{r}_1,t;\mathbf{r}_2,0)}{\partial t} = -e^{-\beta t}\frac{\hbar^2}{2m}\Delta_{\mathbf{r}_1}G_0(\mathbf{r}_1,t;\mathbf{r}_2,0). \tag{36}$$

The solution of equation (36) must satisfy the initial condition

$$G_0(\mathbf{r}_1,0;\mathbf{r}_2,0) = \delta(\mathbf{r}_1 - \mathbf{r}_2).$$

Introducing the modified time t^* by the relation

$$t^* = \frac{1-e^{-\beta t}}{\beta}, \tag{37}$$

we transform equation (36) into the form:

$$i\hbar\frac{\partial G_0(\mathbf{r}_1,t^*;\mathbf{r}_2,0)}{\partial t^*} = -\frac{\hbar^2}{2m}\Delta_{\mathbf{r}_1}G_0(\mathbf{r}_1,t^*;\mathbf{r}_2,0). \tag{38}$$

The solution of equation (38) is well-known and is expressed by the relation

$$G_0(\mathbf{r}_1,t^*;\mathbf{r}_2,0) = (\frac{m}{2\pi i\hbar t^*})^{\frac{3}{2}}e^{i\frac{m}{\hbar}\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{t^*}}. \tag{39}$$

By considering relations (35) and (39), relation (28) has the following form:

$$\begin{aligned} <\mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_a> = \int\int d\mathbf{y}d\mathbf{z} \ <\mathbf{y}|\hat{\rho}(t=0)|\mathbf{z}> (\frac{m}{2\pi\hbar t^*})^3 e^{i\frac{m}{2\hbar}\frac{(\mathbf{r}_b-\mathbf{y}-\mathbf{u}(t))^2}{t^*}} \\ e^{-i\frac{m}{2\hbar}\frac{(\mathbf{r}_a-\mathbf{z}-\mathbf{u}(t))^2}{t^*}}e^{\frac{i}{\hbar}e^{\beta t}m\dot{\mathbf{u}}(t).[\mathbf{r}_b-\mathbf{r}_a]}. \end{aligned} \tag{40}$$

For the statistical description of the dynamics of the Brownian particle we have to calculate the average matrix element of the $\hat{\rho}(t)$. For this purpouse we decompose the quantity $\mathbf{u}(t)$ into parts:

$$\mathbf{u}(t) = \mathbf{d}(t) + \mathbf{e}(t) \tag{41}$$

where

$$\mathbf{d}(t) = \frac{1}{m\beta}\int_0^t [1 - e^{-\beta(t-\tau)}]\mathbf{f}_d(\tau)d\tau \tag{42}$$

and

$$\mathbf{e}(t) = \frac{1}{m\beta}\int_0^t [1 - e^{-\beta(t-\tau)}]\mathbf{f}_c(\tau)d\tau. \tag{43}$$

After substituting relation (41) into relation (40), the average matrix element of the $\hat{\rho}(t)$ is expressed by the relation

$$<< \mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_a>> = \int\int d\mathbf{y}d\mathbf{z} < \mathbf{y}|\hat{\rho}(t=0)|\mathbf{z}> (\frac{m}{2\pi\hbar t^*})^3 \times \tag{44}$$
$$\times e^{i\frac{m}{2\hbar}\frac{(\mathbf{r}_b-\mathbf{y}-\mathbf{d}(t))^2-(\mathbf{r}_a-\mathbf{z}-\mathbf{d}(t))^2}{t^*}} e^{i\frac{m}{\hbar}e^{\beta t}\dot{\mathbf{d}}(t).[\mathbf{r}_b-\mathbf{r}_a]} \times$$
$$\times < e^{-i\frac{m}{\hbar}\mathbf{e}(t).\frac{(\mathbf{r}_b-\mathbf{r}_a+\mathbf{z}-\mathbf{y})}{t^*}+i\frac{m}{\hbar}e^{\beta t}\dot{\mathbf{e}}(t).[\mathbf{r}_b-\mathbf{r}_a]}>.$$

For the averaging of the last term on the right-hand side of relation (44) we shall use the Kubo formula [13]. If the collision force $\mathbf{f}_c(t)$ is defined by the continual Gaussian distribution with the first two statistical moments expressed by relations (14) and (17) then we can write

$$< e^{-i\frac{m}{\hbar}\mathbf{e}(t).\frac{(\mathbf{r}_b-\mathbf{r}_a+\mathbf{z}-\mathbf{y})}{t^*}+i\frac{m}{\hbar}\dot{\mathbf{e}}(t).(\mathbf{r}_b-\mathbf{r}_a)}>= \tag{45}$$
$$= e^{-\frac{m^2D}{\hbar^2}\{\psi(t)\frac{(\mathbf{r}_b-\mathbf{r}_a+\mathbf{z}-\mathbf{y})^2}{t^{*2}}-2e^{\beta t}\varphi(t)\frac{(\mathbf{r}_b-\mathbf{r}_a+\mathbf{z}-\mathbf{y}).(\mathbf{r}_b-\mathbf{r}_a)}{t^*}+e^{2\beta t}\phi(t)(\mathbf{r}_b-\mathbf{r}_a)^2\}}$$

where

$$\psi(t) = \int_0^t \frac{1-e^{-\beta\tau_1}}{\beta}\int_0^t \frac{1-e^{-\beta\tau_2}}{\beta}W(|\tau_1-\tau_2|)d\tau_1 d\tau_2 \tag{46}$$

$$\varphi(t) = \int_0^t \frac{1-e^{-\beta\tau_1}}{\beta}\int_0^t e^{-\beta\tau_2}W(|\tau_1-\tau_2|)d\tau_1 d\tau_2 \tag{47}$$

$$\phi(t) = \int_0^t e^{-\beta\tau_1}\int_0^t e^{-\beta\tau_2}W(|\tau_1-\tau_2)d\tau_1 d\tau_2. \tag{48}$$

3. Analysis of the nondiagonal average matrix element of the $\hat{\rho}(t)$

In this section we shall show that the nondiagonal average matrix elements of the $\hat{\rho}(t)$ damp very fast. For proving this fact we arrange relations (46)-(48) integration by parts and after integration we obtain the following relations:

$$\psi(t) = \frac{2}{\beta^2}\int_0^t (t-\tau)W(\tau)d\tau -$$
$$- \frac{2}{\beta^3}\int_0^t W(\tau)d\tau + \frac{2}{\beta^3}e^{-\beta t}\int_0^t [1+e^{\beta\tau}+ch\beta(t-\tau)]W(\tau)d\tau \tag{49}$$

$$\varphi(t) = \frac{1}{\beta^2}\int_0^t [1-e^{-\beta(t-\tau)}+e^{-2\beta t+\beta\tau}-e^{-\beta t}]W(\tau)d\tau \tag{50}$$

$$\phi(t) = \frac{2}{\beta}e^{-\beta t}\int_0^t sh\beta(t-\tau)W(\tau)d\tau. \tag{51}$$

In asymptotic form ($\beta t \gg 1$) relations (49)-(51) pass into the forms

$$\psi(t) \to a = \frac{2}{\beta^2}\int_0^t (t-\tau)W(\tau)d\tau \tag{52}$$

$$\varphi(t) \to b = \frac{1}{\beta^2}\int_0^t W(\tau)d\tau \tag{53}$$

$$\phi(t) \to c = \frac{1}{\beta}\int_0^t e^{\beta\tau}W(\tau)d\tau. \tag{54}$$

For the δ-correlation function ($W(\tau) = \delta(\tau)$) relations (52)-(54) have the following forms

$$\psi(t) \to \frac{1}{\beta^2}t \tag{55}$$

$$\varphi(t) \to \frac{1}{2\beta^2} \tag{56}$$

$$\phi(t) \to \frac{1}{2\beta}. \tag{57}$$

If we put relations (52)-(54) into relation (45) we shall see that the nondiagonal average matrix element is damped faster than the diagonal average matrix element of the $\hat{\rho}(t)$ due to the existence of the terms $e^{\beta t}$ and $e^{2\beta t}$ on the right-hand side of relation (45).

4. Generalized quantum Fokker-Planck equation

At first we derive the generalized quantum Fokker-Planck equation without assumption about the distribution function of the random force $\mathbf{f}_c(t)$. Then the obtained equation will be considered in the case when the distribution function of the $\mathbf{f}_c(t)$ is Gaussian with the first two moments expressed by relations (14) and (17) or in the case of the δ-correlation function.

According to relation (40), the diagonal element of the $\hat{\rho}(t)$ is expressed by the relation

$$< \mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_b >= e^{-\mathbf{e}(t).\nabla_{\mathbf{r}_b}}e^{-\mathbf{d}(t).\nabla_{\mathbf{r}_b}} \times$$
$$\int\int d\mathbf{y}d\mathbf{z} < \mathbf{y}|\hat{\rho}(t=0)|\mathbf{z}> (\frac{m}{2\pi\hbar t^*})^3 e^{i\frac{m}{2\hbar}\frac{(\mathbf{r}_b-\mathbf{y})^2-(\mathbf{r}_b-\mathbf{z})^2}{t^*}}. \tag{58}$$

Introducing the notation

$$P(\mathbf{r}_b, t) =<< \mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_b>>$$

it can be shown that the average probability distribution $P(\mathbf{r}_b, t)$ obeys the following differential equation

$$\frac{\partial P(\mathbf{r}_b,t)}{\partial t} = - < \dot{\mathbf{e}}(t).\nabla_{\mathbf{r}_b} < \mathbf{r}_b|\hat{\rho}(t)|\mathbf{r}_b>> - \dot{\mathbf{d}}(t).\nabla_{\mathbf{r}_b}P(\mathbf{r}_b,t) + e^{-\beta t}\frac{\partial P(\mathbf{r}_b,t)}{\partial t^*}. \tag{59}$$

Further on we shall assume that

$$< \mathbf{f}_c(t) >= 0.$$

Then using the Kubo formula [13] we can write

$$- < \dot{\mathrm{e}}(t).\nabla_{\mathbf{r}_b} e^{-\mathrm{e}(t).\nabla_{\mathbf{r}_b}} >= \frac{d}{dt'}\frac{d}{d\alpha} < e^{-[\alpha \mathrm{e}(t')+\mathrm{e}(t)].\nabla_{\mathbf{r}_b}} > |_{\alpha=0 t=t'} =$$
$$= \frac{d}{dt'}\frac{d}{d\alpha} e^{\sum_{n=2}^{\infty}(-1)^n \frac{1}{(n-1)!}\alpha \nabla_{\mathbf{r}_b} <\mathrm{e}(t')[\mathrm{e}(t).\nabla_{\mathbf{r}_b}]^{n-1}>_c}|_{\alpha=0 t=t'} \times$$
$$< e^{-\mathrm{e}(t).\nabla_{\mathbf{r}_b}} >= \hat{R} < e^{-\mathrm{e}(t).\nabla_{\mathbf{r}_b}} > \quad (60)$$

where $<>_c$ means the cumulant average.

$$\hat{R} = \sum_{n=2}^{\infty}(-1)^n \frac{1}{(n-1)!}\nabla_{\mathbf{r}_b} < \dot{\mathrm{e}}(t)[\mathrm{e}(t).\nabla_{\mathbf{r}_b}]^{n-1} >_c . \quad (61)$$

After introducing relation (60) into equation (59) we obtain the generalized Fokker-Planck equation:

$$\frac{\partial P(\mathbf{r}_b,t)}{\partial t} = (\hat{R} - \dot{\mathbf{d}}(t).\nabla_{\mathbf{r}_b})P(\mathbf{r}_b,t) + e^{-\beta t}\frac{\partial P(\mathbf{r}_b,t)}{\partial t^*}. \quad (62)$$

Equation (62) is valid for the arbitrary distribution function of the random force $\mathbf{f}_c(t)$. When the distribution function is Gaussian with the first two moments expressed by relations (14) and (17) then

$$\hat{R} = 2D\varphi(t)\Delta_{\mathbf{r}_b} \quad (63)$$

and for δ-correlation function according to relation (56)

$$\hat{R} = \frac{D}{\beta^2}\Delta_{\mathbf{r}_b} \quad (64)$$

for $(\beta t \gg 1)$.

In the last case equation (62) is the standard Fokker-Planck equation because $t^* \to \frac{1}{\beta}$.

5. Conclusion

In this paper we have obtained the following results:

The average matrix element of the statistical operator $\hat{\rho}(t)$ has been calculated in the case of the Gaussian distribution function of the random force $\mathbf{f}_c(t)$.

We have found out that the nondiagonal average matrix element is damped faster than the diagonal average element of the $\hat{\rho}(t)$.

We have derived the generalized Fokker-Planck equation valid for the arbitrary distribution function of the random force $\mathbf{f}_c(t)$.

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