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NEW EVALUATION OF HADRONIC CONTRIBUTIONS TO THE ANOMALOUS MAGNETIC MOMENT OF CHARGED LEPTONS

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A re-evaluation of the lowest-order hadronic vacuum-polarization contribution to the anomalous magnetic moment of the electron, muon and tau-lepton with a higher precision in comparison with previous estimates is carried out. The latter is achieved because new data on some exclusive processes have appeared recently, more accomplished models for a description of the pion and kaon electromagnetic structure have been developed and the revised (due to a new value of the coefficient of the third power of α_s) QCD formula for R=$\sigma_{tot}(e^+e^- \to had)/\sigma_{tot}(e^+e^- \to \mu^+\mu^-)$ with electroweak corrections has been applied to analyze all existing data in a proper way. The final results are $a_e^{(2)had} = (1.810 \pm 0.011 \pm 0.002) \times 10^{-12}$, $a_\mu^{(2)had} = (6.986 \pm 0.042 \pm 0.016) \times 10^{-8}$ and $a_\tau^{(2)had} = (3.436 \pm 0.024 \pm 0.024) \times 10^{-6}$.

1. Introduction

Since the charged leptons obey the Dirac equation, their magnetic moments are related to the spin by

$$\mu_\ell = g_\ell \frac{e}{2m_\ell c} \cdot \frac{\hbar}{2}, \tag{1}$$

where $g_\ell = 2$ theoretically. The g-factor reflects the point-like nature of the Dirac equation and deviations from this expected value are of great interest since they would indicate a substructure of the lepton under consideration. However, even in the absence of any intrinsic structure the interactions existing in nature modify the g-factor because of the emission and absorption of virtual photons (electromagnetic effects), intermediate vector bosons (weak interaction effects) and the vacuum polarization into virtual hadronic states (strong interaction effects).

It has become conventional to describe the modification of the g-factor by the magnetic anomaly defined by

$$a_\ell \equiv \frac{g_\ell - 2}{2} = a_\ell^{(1)} \cdot \frac{\alpha}{\pi} + (a_\ell^{(2)QED} + a_\ell^{(2)had}) \cdot (\frac{\alpha}{\pi})^2 + a_\ell^{(2)weak} + O(\frac{\alpha}{\pi})^3, \tag{2}$$

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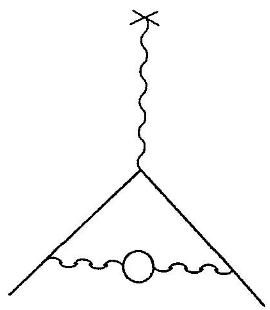

Fig. 1. The lowest-order hadronic vacuum polarization contribution to the anomalous magnetic moment of charged leptons $l = e, \mu, \tau$.

where $\alpha \approx 1/137$ is a fine structure constant.

In this paper a new more precise evaluation of the lowest-order hadronic vacuum-polarization contribution $a_\ell^{(2)had}(\ell = e, \mu, \tau)$ given by the Feynman diagram of Fig. 1. is presented. Reasons why it is worth while to realize this program are as follows.

There is a new $g-2$ muon experiment [1] under way at Brookhaven National Laboratory expected to be performed with an accuracy of

$$\Delta a_\mu^{\exp} = \pm 0.05 \times 10^{-8}, \tag{3}$$

which could represent a good test of the standard electroweak model provided that the hadronic part can be determined sufficiently precisely.

Because of precise QED calculations up to four loops [2], yielding

$$a_\mu^{QED} = (116584.80 \pm 0.03) \times 10^{-8} \tag{4}$$

and the one loop weak interaction contribution evaluation [3] as

$$a_\mu^{(2)weak} = 0.181 \times 10^{-8}, \tag{5}$$

as well as the highly accurate measurement [4] of the latest experimental value

$$a_\mu^{\exp} = (116592.30 \pm 0.84) \times 10^{-8} \tag{6}$$

the lowest order hadronic vacuum-polarization contribution $a_\mu^{(2)had}$ (dominating among all other strong interaction contributions) is by far not negligible. The recent evaluations [2],[5],[6] show

$$a_\mu^{had} = (7.070 \pm 0.060 \pm 0.170) \times 10^{-8} \quad \text{(KNO)[2]} \tag{7}$$

$$a_\mu^{had} = (7.100 \pm 0.105 \pm 0.049) \times 10^{-8} \quad \text{(CLY)[5]} \tag{8}$$

$$a_\mu^{had} = (7.052 \pm 0.060 \pm 0.046) \times 10^{-8} \quad \text{(MD)[6]} \tag{9}$$

that are about eight times of the experimental uncertainty in (6). From the same results it is straightforward to see that the up to now most precise evaluation of $a_\mu^{(2)had}$

is achieved by Martinovič and Dubnička[6] with a total error to be about one half of the one loop weak interaction contribution (5), while total errors of two previous evaluations [3],[5] are just comparable with (5).

In the next sections of this paper we show that owing to

- inclusion into the analysis of additional data on some exclusive processes (K^+K^-[7], $K^0\bar{K}^0$[8], 3π[9], $\omega\pi^0$[9] and 4π[9])
- application of more accomplished models [10] for a description of the pion and kaon electromagnetic structure
- a more correct calculation of the external covariance matrix in an estimation of errors in 2π and $K\bar{K}$ channels
- application of the revised QCD formula [11],[12] for $R = \sigma_{tot}(e^+e^- \to had)/\sigma_{tot}(e^+e^- \to \mu^+\mu^-)$ with electroweak corrections [13] in order to evaluate a contribution to $a_\ell^{(2)had}$ from the high-energy region in a proper way

we are able to diminish the error $\Delta a_\mu^{(2)had}$ to be approximately four times less than the one loop weak interaction contribution (5) and thus it becomes comparable with accuracy (3) expected in the new $g-2$ muon experiment at Brookhaven National Laboratory.

Since the procedure of evaluation of $a_e^{(2)had}$ and $a_\tau^{(2)had}$ is very similar to the evaluation of $a_\mu^{(2)had}$, we obtain new results on the latter quantities too.

The next section is devoted to a short review of the formalism used in the evaluation $a_\ell^{(2)had}(\ell = e, \mu, \tau)$. Our practical treatment of $a_\ell^{(2)had}$ and the corresponding error analysis is given in Sect. 3, while conclusions and summary are given in Sect. 4.

2. The lowest-order hadronic vacuum-polarization contribution to a_ℓ

All evaluations of $a_\ell^{(2)had}(\ell = e, \mu, \tau)$ are based on the integral representation [14]

$$a_\ell^{(2)had} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} \sigma^h(s) K_\ell(s) ds \tag{10}$$

where $\sigma^h(s)$ stands for the total cross section $\sigma(e^+e^- \to had)$ and

$$K_\ell(s) = \int_0^1 \frac{x^2(1-x)}{x^2 + (1-x)s/m_\ell^2} dx. \tag{11}$$

The integral (11) can be calculated explicitly and its final form depends on the mass of a lepton under consideration in relation to the pion mass m_π.

Since m_e, $m_\mu < m_\pi$, then for the electron and muon one obtains

$$K_{e,\mu} = \left(\frac{1}{2} - A\right) - \left(A - \frac{A^2}{2}\right) \ln A - \frac{A(A^2 - 4A + 2)}{2\sqrt{A^2 - 4A}} \ln \frac{[(2-A) - \sqrt{a^2 - 4A}][-A + \sqrt{a^2 - 4A}]}{[(2-A) + \sqrt{a^2 - 4A}][-A - \sqrt{a^2 - 4A}]} \tag{12}$$

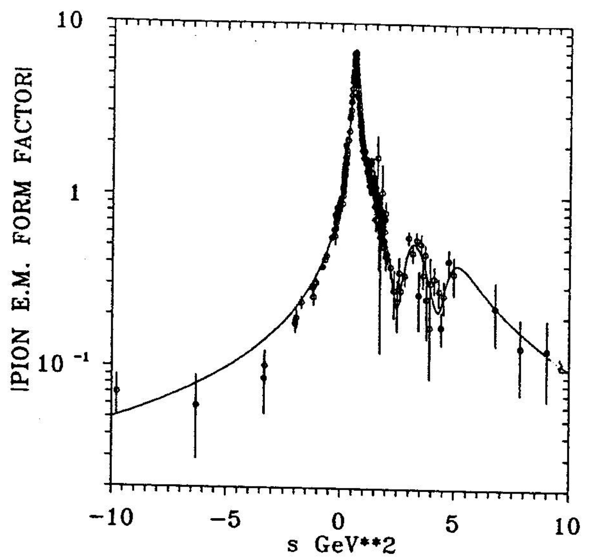

Fig. 2. The up to now most accomplished reproduction of all existing pion e.m ff data by means of the unitary and analytic VMD model (20).

for all s from the range $4m_\pi^2 < s < \infty$, where $A = s/m_{e,\mu}^2$.

On the other hand, as $m_\tau > m_\pi$, for the τ-lepton the explicit form of the integral (11) depends on the relation of s to $4m_\tau^2$.

If $4m_\pi^2 < s < 4m_\tau^2$, then

$$K_\tau(s) = (\frac{1}{2} - A) - (A - \frac{A^2}{2})\ln A - \frac{A(A^2-4A+2)}{\sqrt{4A-A^2}}\left[\arctan\frac{2-A}{\sqrt{4A-A^2}} - \arctan\frac{-A}{\sqrt{4A-A^2}}\right] \tag{13}$$

and for all $s > 4m_\tau^2$ the $K_\tau(s)$ has an identical form with (12), where now $A = s/m_\tau^2$.

At this place we would like to note that equation (7) in ref.[6] and expression (A.3) in the second paper of ref.[2], even the explicit formula in ref.[15], are not completely correct, however their influence upon the final result is negligible.

So, the most important component in the evaluation of $a_\ell^{(2)had}$ is a rich experimental information on the total cross section $\sigma^h(s)$ from threshold to high energies employed either in the form of experimental points with errors or in the form of models used in a perfect description of the cross section $\sigma^h(s)$. Sometimes both of these approaches are taken in order to be complementary.

From the practical point of view it is advantageous to divide the integral in (10) into the low-energy ($4m_\pi < s < s_0$) and high-energy parts ($s_0 < s < +\infty$) and to write it in the form

$$a_\ell^{(2)had} = \frac{1}{4\pi^3}\left\{\int_{4m_\pi^2}^{s_0}\left[\sum_F \sigma^h(e^+e^- \to F)\right]K_\ell(s)ds + \int_{s_0}^{\infty} \mathrm{R}(e^+e^- \to had)\sigma_{tot}(e^+e^- \to \mu^+\mu^-)K_\ell(s)ds\right\}, \tag{14}$$

where we left s_0 to vary in the range $2\mathrm{GeV}^2 < s_0 < 4\ \mathrm{GeV}^2$ to demonstrate the independence of the final results on the choice of s_0. The sum in (14) goes over all known exclusive final states.

The variation of s_0 was enabled by the existence of an overlap of the data on the total cross sections of exclusive processes $e^+e^- \to F$ and the data on $\mathrm{R}=\sigma_{tot}(e^+e^- \to had)/\sigma_{tot}(e^+e^- \to \mu^+\mu^-)$. Finally it provides a reliability test of an evaluation of (14).

We note before a presentation of the numerical results, that such a variation of s_0 gives, in the most interesting from the experimental point of view case of the muon, the change of the central value less than one half of the estimated total error $\Delta a_\mu^{(2)had}$.

The central value given by (10) can be determined in two ways. The function $K_\ell(s)$ can be taken in the explicit form (12) or (13), then we carry out a numerical integration only over one integral from $4m_\pi^2$ to ∞, or $K_\ell(s)$ is taken in the integral form (11), then we carry out a numerical integration over two integrals simultaneously. Practically we have verified that both ways lead to the same results if sufficient precision in numerical integration is attained.

In order to achieve realistic and quantitative error estimates, we have included in our analysis the uncertainties coming from the experimental input as well as the ones induced by the models used for the cross section $\sigma^h(s)$. Then the total error $\Delta a_\ell^{(2)had}$ consists of three parts as follows

$$\Delta a_\ell^{(2)had} = \sqrt{[\Delta a_\ell^{(2)had}(\mathrm{stat})]^2 + [\Delta a_\ell^{(2)had}(\mathrm{syst})]^2 + [\Delta a_\ell^{(2)had}(\mathrm{mod})]^2} \tag{15}$$

where $\Delta a_\ell^{(2)had}(\mathrm{stat})$ is a statistical error, $\Delta a_\ell^{(2)had}(\mathrm{syst})$ is a systematic error and $\Delta a_\ell^{(2)had}(\mathrm{mod})$ is the model error defined as a deviation of the value of a contribution obtained by integration over experimental points using the trapezoidal rule and by the integration based on the model parametrization. All $\Delta a_\ell^{(2)had}(\mathrm{stat})$, $\Delta a_\ell^{(2)had}(\mathrm{syst})$ and $\Delta a_\ell^{(2)had}(\mathrm{mod})$ consist of sums of the corresponding errors evaluated in all exclusive channels, including also the high energy region contribution determined by R. Here we meet two types of situations. First, $\sigma^h(s)$ (or R) is given by data with errors and simultaneously we have a well founded model for a description of the latter. In this case we evaluate the error to be compound of statistical and systematic errors, and independently we evaluate the model error. Second, $\sigma^h(s)$ is given only by data with errors i.e. there is no satisfactory model able to describe the latter. In this case we evaluate only statistical and systematical errors.

3. Numerical evaluation of contributions from exclusive channels and high energy region

First, we concentrate on the evaluation of contributions to $a_\ell^{(2)had}$ from the low-energy region given by the first integral in (14), where the sum contains total cross sections of exclusive processes as follows

$$\begin{aligned}\sum_F \sigma^h(e^+e^- \to F) &= \sigma_{tot}(e^+e^- \to \pi^+\pi^-) + \sigma_{tot}(e^+e^- \to K^+K^-) + \\ &+ \sigma_{tot}(e^+e^- \to K^0\bar{K}^0) + \sigma_{tot}(e^+e^- \to \pi^+\pi^-\pi^0) + \\ &+ \sigma_{tot}(e^+e^- \to \omega\pi^0) + \sigma_{tot}(e^+e^- \to \pi^+\pi^-2\pi^0) + \\ &+ \sigma_{tot}(e^+e^- \to 2\pi^+2\pi^-) + \sigma_{tot}(e^+e^- \to 2\pi^+2\pi^-\pi^0) + \\ &+ \sigma_{tot}(e^+e^- \to 3\pi^+3\pi^-).\end{aligned} \tag{16}$$

All other known exclusive channels are not considered here because they give negligible contributions on the level of estimated errors. The latter is caused by two reasons. First, the cross sections of channels like $K^0K^\pm\pi^\pm$, $K^+K^-\pi^0$, $K^+K^-\pi^+\pi^-$ are of order 1 - 3 nb. Second, as can be seen from Eq.(11), $K_\ell(s)$ behaves like $\approx m_\ell^2/s$ for $s >> m_\ell^2$, suppressing in this way the contributions from the higher energy region.

Now we treat each considered channel in (16) in detail.

2π : According to the preceding remark it will be the contribution of the process $e^+e^- \to \pi^+\pi^-$ which will dominate in $a_\ell^{(2)had}$ for all (see Tables 1-3 below) $\ell = e, \mu$ and τ. Its cross section is given by

$$\sigma^{2\pi}(s) = \frac{\pi\alpha^2\beta_\pi^3}{3s}|F_\pi^{I=1}(s) + Re^{i\phi}\frac{m_\omega^2}{m_\omega^2 - s - im_\omega\Gamma_\omega}|^2 \tag{17}$$

where $\beta_\pi = (1 - 4m_\pi^2/s)^{1/2}$ is the velocity of an outgoing pion in the c.m. system and R and ϕ are the $\rho - \omega$ interference amplitude and phase, respectively. A diminishing of estimated errors in comparison with previous evaluations [6] was achieved by a more accomplished unitary and analytic VMD model [10] of $F_\pi^{E,I=1}(s)$ with three excited states $\rho'(1450)$, $\rho''(1700)$, $\rho'''(2150)$ of the $\rho(770)$ meson which reproduces all existing pion electromagnetic (e. m.) form factor (ff) data more precisely (see model errors in Tables 1-3.) It is obtained by the incorporation of a two-cut approximation of the correct pion e.m. ff analytic properties into the standard VMD model defined by the relation

$$F_\pi^{E,I=1}(s) = \sum_{v=\rho,\rho',\rho'',\rho'''} \frac{m_v^2}{m_v^2 - s}\frac{f_{v\pi\pi}}{f_v}. \tag{18}$$

Practically, the latter is performed [10] by application of the nonlinear transformation

$$s = s_0 - \frac{4(s_1 - s_0)}{(1/W - W)^2} \tag{19}$$

Table 1. Partial contributions to $a_e^{(2)had}$

Channel	Central value $\times 10^{-12}$	Exp. error $\times 10^{-15}$	Model error $\times 10^{-15}$
$\pi^+\pi^-$	1.3610	5.614	0.31
K^+K^-	0.0593	2.386	1.42
$K_S^0K_L^0$	0.0325	1.592	1.42
$\pi^+\pi^-\pi^0$	0.1350	6.769	-
$\omega\pi^0$	0.0187	1.719	-
$\pi^+\pi^-\pi^0\pi^0$	0.0336	5.080	-
$\pi^+\pi^-\pi^+\pi^-$	0.0147	1.534	-
5π	0.0008	0.247	-
6π	0.00003	0.064	-
QCD 1.reg. $2\ \mathrm{GeV}^2 \le s \le 9.61\ \mathrm{GeV}^2$	0.0633		0.002
QCD 2.reg. $20.2\ \mathrm{GeV}^2 \le s \le 81.0\ \mathrm{GeV}^2$	0.0208	0.974	0.003
QCD 3.reg. $s \ge 196.0\ \mathrm{GeV}^2$	0.0248		0.004
$c\bar{c}$ reg. $9.61\ \mathrm{GeV}^2 \le s \le 20.2\ \mathrm{GeV}^2$	0.0215	0.581	-
$b\bar{b}$ reg. $81.0\ \mathrm{GeV}^2 \le s \le 196.0\ \mathrm{GeV}^2$	0.0045	0.025	-
Resonances	0.0197	1.147	-

to (18) with subsequent correct incorporation of the nonzero values of vector-meson widths taking into account at the same time the relations

$$\begin{aligned}(m_\rho^2 - \Gamma_\rho^2/4) < s_1; \qquad (m_{\rho'}^2 - \Gamma_{\rho'}^2/4) > s_1 \\ (m_{\rho''}^2 - \Gamma_{\rho''}^2/4) > s_1; \qquad (m_{\rho'''}^2 - \Gamma_{\rho'''}^2/4) > s_1\end{aligned} \tag{20}$$

found in a fitting procedure, which all together lead finally to the unitary and analytic VMD model of pion e.m. structure

$$F_\pi^{E,I=1}(s) = \left(\frac{1 - W^2}{1 - W_N^2}\right)^2 \cdot \left[\frac{(W_N - W_\rho)(W_N - W_\rho^*)(W_N - 1/W_\rho)(W_N - 1/W_\rho^*)}{(W - W_\rho)(W - W_\rho^*)(W - 1/W_\rho)(W - 1/W_\rho^*)}(f_{\rho\pi\pi}/f_\rho) + \sum_{v=\rho',\rho'',\rho'''}\frac{(W_N - W_v)(W_N - W_v^*)(W_N + W_v)(W_N + W_v^*)}{(W - W_v)(W - W_v^*)(W + W_v)(W + W_v^*)}(f_{v\pi\pi}/f_v)\right] \tag{21}$$

The parameters in (19) $s_0 = 4m_\pi^2$ and $s_1 \approx 1\mathrm{GeV}^2$ (in a fit of the data s_1 was left to be free parameter) are square-root branch points generating a four-sheeted Riemann surface in s-variable on which the model (21) is defined.

Table 2. Partial contributions to the $a_\mu^{(2)had}$

Channel	Central value $\times 10^{-11}$	Exp. error $\times 10^{-11}$	Model error $\times 10^{-11}$
$\pi^+\pi^-$	4942.16	20.59	0.86
K^+K^-	232.53	9.34	5.54
$K_S^0 K_L^0$	127.31	6.23	5.54
$\pi^+\pi^-\pi^0$	512.36	25.57	-
$\omega\pi^0$	74.15	6.74	-
$\pi^+\pi^-\pi^0\pi^0$	133.82	20.23	-
$\pi^+\pi^-\pi^+\pi^-$	59.99	10.45	-
5π	3.17	0.99	-
6π	1.30	0.44	-
QCD 1.reg. $2\ \mathrm{GeV}^2 \le s \le 9.61\ \mathrm{GeV}^2$	604.78		8.22
QCD 2.reg. $20.2\ \mathrm{GeV}^2 \le s \le 81.0\ \mathrm{GeV}^2$	89.98	2.00	11.76
QCD 3.reg. $s \ge 196.0\ \mathrm{GeV}^2$	10.70		1.68
$c\bar{c}$ reg. $9.61\ \mathrm{GeV}^2 \le s \le 20.2\ \mathrm{GeV}^2$	92.48	2.40	-
$b\bar{b}$ reg. $81.0\ \mathrm{GeV}^2 \le s \le 196.0\ \mathrm{GeV}^2$	19.40	0.22	-
Resonances	83.03	4.83	-

A perfect reproduction of all existing data on the pion e.m. ff by means of (21), where also the isospin violating $\omega \to \pi^+\pi^-$ decay contribution (the so-called $\rho - \omega$ interference effect) was taken into account by means of (17) with R as an additional free parameter and the phase to be [16]

$$\phi = \arctan \frac{m_\rho \Gamma_\rho}{(m_\rho^2 - m_\omega^2)} \tag{22}$$

is presented in Fig.2.

Another diminishing of estimated errors in this case in comparison with previous evaluation [6] was achieved by a more profound calculation of the external covariance matrix $C_{i,j}$ (as given by the HESSE subroutine of the MINUIT program with UP adjusted to the number of fitted parameters) determining the experimental error to be given by

$$\sigma^2 = \sum_{i,j} C_{i,j} D_i D_j \tag{23}$$

where $D_i = \partial a_\ell / \partial A_i$ and A_i are fitted parameters. The parameters A_i have no upper and lower limits in finding the definitive minimum. The latter leads to $C_{i,j}$, which gives smaller error (23) than the approach with limited parameter ranges.

The evaluated 2π channel contributions to $A_\ell^{(2)had}$ for $l = e, \mu$ and τ are numerically presented in the first lines of Tables 1-3.

$2K$: In these channels we have found older data of Novosibirsk on the $e^+e^- \to K^+K^-$ and $e^+e^- \to K^0\bar{K}^0$ total cross sections in the ϕ-region measured in arbitrary units. Since the value of the latter in both charged and neutral cases is known at the maximum of the peak we were able to scale all other experimental points.

There are also new DM2 data [17] on the $e^+e^- \to K^+K^-$ total cross section which have been taken into account in the analysis of all existing data by means of the unitary and analytic VMD model of the e.m. structure of kaons unlike the paper of ref.[6]. Of course a more correct calculation of the internal covariance matrix $C_{i,j}$, like in the 2π case was carried out.

As a result (see Table 2), smaller values of $K\bar{K}$ contributions to $a_\mu^{(2)had}$ in comparison with [6] were obtained, however, with a remarkable diminishing of experimental errors.

$\pi^+\pi^-\pi^0$: Unlike the previous calculations in ref. [6], new data on $\sigma_{tot}(e^+e^- \to 3\pi)$ were published [9]. As a consequence the resonant model formula in [6] is no more applicable. Therefore a contribution of 3π channel to $a_\ell^{(2)had}$ was evaluated only numerically by means of the trapezoidal integration over experimental points. One can see immediately from Tables 1 - 3 that the 3π channel contribution is in magnitude on the second place.

$\omega\pi^0$: This channel was not taken into account explicitly in ref. [6], because at that time it was contained in global data on $\pi^+\pi^-\pi^0\pi^0$ channel. Now separate data on $\sigma_{tot}(e^+e^- \to \omega\pi^0)$ appeared [9] and corrected data on $\sigma_{tot}(e^+e^- \to \pi^+\pi^-\pi^0\pi^0)$ were published [9]. The contribution of $\omega\pi^0$ channel to $a_\ell^{(2)had}$ was evaluated also by the trapezoidal integration directly over experimental points.

$4\pi, 5\pi$ and 6π : All these channels were completed by new data and contributions of them to $a_\ell^{(2)had}$ were evaluated by the numerical integration. Results are presented in Tables 1 - 3.

Now we are left with high energy contributions to $a_\ell^{(2)had}$. They are obtained by means of the second integral in (14). For parametrization of $R(e^+e^- \to had)$ we use the corrected QCD formula up to the third order of α_s, where the coefficient of the third power of α_s is changed from the value (+64) used in ref. [6] to the value (-12) (see ref. [11],[12]). Moreover, the electroweak corrections are taken into account in the sense of the Marshall analysis [13], but in our case all electroweak parameters are fixed at the table values and only the scale parameter of QCD $\Lambda_{\overline{MS}}$ is left to be a free parameter.

The whole high energy region is divided into three regions, the natural boundaries of which are thresholds of the creation of $q\bar{q}$ pairs as it is shown in Tables 1 -3. In the description of data on $R(e^+e^- \to had)$ all resonances are formally disregarded and their contributions to $a_\ell^{(2)had}$ were evaluated separately. Then all three regions like in ref. [6],[13] are described by the QCD corrected formula of $R(s)$ with the number of quarks $n_f = 3$, $n_f = 4$ and $n_f = 5$, respectively. Every region is thus characterized by a different value of $\Lambda_{\overline{MS}}^{(i)}$ $i = 3, 4, 5$. Then the scale parameter of QCD Λ_{MS} is identified with $\Lambda_{\overline{MS}}^{(5)}$, that upon our fitting procedure takes the value

$$\Lambda_{\overline{MS}} = 398 \pm 45\mathrm{MeV}. \tag{24}$$

The contributions of all resonances existing up to now in the region described by $R(s)$ were evaluated like in ref. [6] in the narrow-width approximation.

Table 3. Partial contributions to the $a_\tau^{(2)had}$

Channel	Central value $\times 10^{-8}$	Exp. error $\times 10^{-8}$	Model error $\times 10^{-8}$
$\pi^+\pi^-$	170.10	0.73	0.38
K^+K^-	12.68	0.50	0.84
$K_S^0 K_L^0$	6.80	0.33	0.84
$\pi^+\pi^-\pi^0$	21.84	1.08	-
$\omega\pi^0$	4.62	0.42	-
$\pi^+\pi^-\pi^0\pi^0$	8.62	1.30	-
$\pi^+\pi^-\pi^+\pi^-$	3.93	1.01	-
5π	0.21	0.06	-
6π	0.11	0.04	-
QCD 1.reg. $2\ \mathrm{GeV}^2 \le s \le 9.61\ \mathrm{GeV}^2$	59.72		1.04
QCD 2.reg. $20.2\ \mathrm{GeV}^2 \le s \le 81.0\ \mathrm{GeV}^2$	17.72	0.40	1.84
QCD 3.reg. $s \ge 196.0\ \mathrm{GeV}^2$	2.79		0.42
$c\bar{c}$ reg. $9.61\ \mathrm{GeV}^2 \le s \le 20.2\ \mathrm{GeV}^2$	14.66	0.36	-
$b\bar{b}$ reg. $81.0\ \mathrm{GeV}^2 \le s \le 196.0\ \mathrm{GeV}^2$	4.64	0.05	-
Resonances	12.54	0.71	-

The summation of all contributions in Tables 1-3 give the following final values

$$a_e^{(2)had} = (1.810 \pm 0.011 \pm 0.002) \times 10^{-12} \quad (25)$$

$$a_\mu^{(2)had} = (6.986 \pm 0.042 \pm 0.016) \times 10^{-8} \quad (26)$$

$$a_\tau^{(2)had} = (3.436 \pm 0.024 \pm 0.024) \times 10^{-6} \quad (27)$$

They are up to now the most precise determined values of the lowest-order hadronic vacuum polarization contributions to the anomalous magnetic moment of charged leptons.

4. Conclusions and summary

By using the standard procedure of evaluation of the lowest order hadronic vacuum-polarization contribution $a_\ell^{(2)had}$ to the anomalous magnetic moment of leptons based on the integral representation (10) we have diminished total errors of the latter for all charged leptons. This was achieved due to inclusion of additional data on some exclusive processes, application of more accomplished models for the description of the pion and kaon electromagnetic structure, a more correct calculation of the external covariance matrix as given by the HESSE subroutine of the MINUIT program and application of the revised value of the coefficient of the third power of QCD formula for R(s).

Comparing (24) with the total value of the anomalous magnetic moment of the electron [18]

$$a_e = (1159652140 \pm 28) \times 10^{-12} \quad (28)$$

we come to the conclusion that our precision given by the errors in (25) is not at the center of interest.

A different situation is in the case of the τ-lepton, where only a rough estimate [19]

$$a_\tau^{(2)had} = (3.6 \pm 0.3 \pm 0.1) \times 10^{-6} \quad (29)$$

was up to now carried out in contrast to our precise results (27).

In the very actual case of the μ-meson we have achieved diminishing of the error $\Delta a_\mu^{(2)had}$ down to one fourth (see (26)) of the one loop weak interaction contribution (5). This accuracy becomes comparable with the accuracy (3) expected in the new $g-2$ muon experiment to be under way at Brookhaven National Laboratory [1].

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